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Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime

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Abstract

Generalized Kerr-NUT-de Sitter spacetime is the most general spacetime which admits a rank-2 closed conformal Killing-Yano tensor. It contains the higher-dimensional Kerr-de Sitter black holes with partially equal angular momenta. We study the separability of gravitational perturbations in the generalized Kerr-NUT-de Sitter spacetime. We show that a certain type of tensor perturbations admits the separation of variables. The linearized perturbation equations for the Einstein condition are transformed into the ordinary differential equations of Fuchs type.

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1 Introduction

The higher-dimensional Kerr-NUT-de Sitter metric was constructed by Chen-Lü-Pope [1]. The metric is the most general known solution describing the rotating asymptotically de Sitter black hole spacetime with NUT parameters. It has been shown in [2, 3] that the Kerr-NUT-de Sitter spacetime has a rank-2 closed conformal Killing-Yano (CKY) tensor [4]. This tensor generates the tower of Killing-Yano and Killing tensors, which implies complete integrability of geodesic equations [5, 6] and complete separation of variables for the Hamilton-Jacobi [2], Klein-Gordon [2] and Dirac equations [7] (for reviews, see [8, 9, 10]). Furthermore, it was proved that the Kerr-NUT de Sitter spacetime is the only spacetime when the eigenvalues of the CKY tensor are functionally independent [11, 12, 13].

Recently, we have obtained the most general metric admitting a rank-2 closed CKY tensor [14, 15]. The CKY tensor generally has the constant eigenvalues besides the functionally independent eigenvalues. Associated with these constant eigenvalues the spacetime admits Kähler manifolds of the same dimension as the multiplicity of them. Then, the metric may be locally written as a Kaluza-Klein metric on the bundle over the Kähler manifolds whose fibers are Kerr-NUT-de Sitter spacetimes. We call such a spacetime the generalized Kerr-NUT-de Sitter spacetime.

Important examples are given by a special class of Kerr-de Sitter metrics found by Gibbons-Lü-Page-Pope [16, 17]. In D dimensions the Kerr-de Sitter metric has $[(D-1)/2]$ angular momenta. When some of them are equal, the CKY tensor has constant eigenvalues. Then, the coordinates used in [1], which are based on the eigenvalues of the CKY tensor, are not effective to express the metric since the constant eigenvalues do not work as coordinates. Actually, such a spacetime belongs not to the Kerr-NUT-de Sitter but to the generalized Kerr-NUT-de Sitter spacetime. In particular the odd-dimensional Kerr-de Sitter metric for which all angular momenta are equal describes a cohomogeneity one spacetime.

In this paper we study the separability of gravitational perturbations in the general-

ized Kerr-NUT-de Sitter spacetime. In four dimensions there exists a master equation describing the perturbations of the rotating black holes [18, 19]. However, in higher-dimensional spacetimes no one has succeeded in finding such an equation. Recently, there was a certain progress in this problem [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. In [23] the perturbation of odd-dimensional Myers-Perry black hole with equal angular momenta was studied. They found a class of perturbations for which the equations of motion reduce to a single radial equation. We extend their analysis to the case of the generalized Kerr-NUT-de Sitter spacetime. Our main result shows that the tensor type gravitational perturbations for the generalized Kerr-NUT-de Sitter spacetime admit the separation of variables.

This paper is organized as follows. In section 2 we briefly describe the properties of the generalized Kerr-NUT-de Sitter metric. In section 3 the solutions of the Einstein condition are briefly reviewed. To illustrate the solutions, we give a subfamily of the solutions which represents the higher-dimensional Kerr-de Sitter black holes with partially equal angular momenta (and with some zero angular momenta). This provides important examples of our formulation. In section 4 we show the separability of the tensor type perturbations. It should be emphasised that the separability is a rather non-trivial consequence of the detailed structure of the spacetime. A recent study of the separability in the gravitational perturbations was restricted to the case of all angular momenta equal, where the spacetime is of cohomogeneity one. In our case, the equations can be separated in the situation where the black holes have angular momenta of plural different values. In section 5 we summarize the results and comment on open questions. In Appendices A and B, we present the connection 1-forms and the curvature 2-forms explicitly. These quantities are essential to our calculations. Some of the results presented here are already available in [14, 15]. However, we attempt to make this paper as self-contained as possible. We also provide Appendices C and D, which contain some details of our calculations. We obtain an explicit coordinate transformation from the higher-dimensional Kerr-de Sitter black hole with partially equal angular momenta to the generalized Kerr-NUT-de Sitter

spacetime. In Appendix E, we present non-zero components of the Lichnerowicz operator.

2 Generalized Kerr-NUT-de Sitter spacetime

In this section, we review the metric on the generalized Kerr-NUT-de Sitter spacetime which admits a rank-2 closed CKY tensor.

We start with the $2n$-dimensional Kerr-NUT-de Sitter metric found by Chen-Lü-Pope [1]. The metric takes the form

$$g^{(2n)} = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu(x)} + \sum_{\mu=1}^{n} Q_\mu(x) \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) d\psi_k \right)^2 , \tag{2.1}$$

where Q_μ is defined by

$$Q_\mu = \frac{Y_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} (x_\mu^2 - x_\nu^2) \tag{2.2}$$

with a function $Y_\mu = Y_\mu(x_\mu)$ depending only on x_μ. The coordinates ψ_k give the Killing vectors $\partial/\partial\psi_k$ $(k = 0, 1, \dots, n-1)$. The symbol $\sigma_k(\hat{x}_\mu)$ are the k-th elementary symmetric functions of $\{x_\nu^2; \nu \neq \mu\}$:

$$\prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} (t - x_\nu^2) = \sigma_0(\hat{x}_\mu) t^{n-1} - \sigma_1(\hat{x}_\mu) t^{n-2} + \cdots + (-1)^{n-1} \sigma_{n-1}(\hat{x}_\mu). \tag{2.3}$$

The spacetime admits a rank-2 non-degenerate closed CKY tensor [2, 3]. Without assuming the non-degeneracy the classification of higher-dimensional spacetimes (M, g) with a rank-2 closed CKY tensor was obtained in [14, 15]. We call such spacetimes the generalized Kerr-NUT-de Sitter spacetimes. They have a bundle structure; the fiber space is the $2n$-dimensional Kerr-NUT-de Sitter spacetime and the base space is a product space

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(N)} \times M^{(0)}, \tag{2.4}$$

where the manifolds $M^{(i)}$ $(i = 1, 2, \dots, N)$ are $2m_i$-dimensional Kähler manifolds with metrics $g^{(i)}$, and $M^{(0)}$ is an $m^{(0)}$-dimensional Riemann manifold with a metric $g^{(0)}$. The

dimension D of the generalized Kerr-NUT-de Sitter spacetime is given by

$$D = 2n + 2|m| + m^{(0)}, \qquad |m| := \sum_{i=1}^{N} m_i. \tag{2.5}$$

The coordinates (x_μ, ψ_k) ($\mu = 1, 2, \ldots, n$ and $k = 0, 1, \ldots, n-1$) are coordinates on the $2n$-dimensional Kerr-NUT-de Sitter fiber space. The fiber metric is twisted by the Kähler forms $\omega^{(i)}$ corresponding to the metric $g^{(i)}$, i.e., the 1-form $d\psi_k$ in (2.1) is replaced by a 1-form θ_k. Locally, $\omega^{(i)} = dA^{(i)}$ and so we can write it as

$$\theta_k = d\psi_k - 2\sum_{i=1}^{N}(-1)^{(n-k)}\xi_i^{2(n-k)-1}A^{(i)}, \qquad k = 0, 1, \ldots, n-1. \tag{2.6}$$

The metric on the generalized Kerr-NUT-de Sitter spacetime is of the form

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu(x)} + \sum_{\mu=1}^{n} P_\mu(x)\left(\sum_{k=0}^{n-1}\sigma_k(\hat{x}_\mu)\theta_k\right)^2 + \sum_{i=1}^{N}\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2)g^{(i)} + \sigma_n g^{(0)}, \tag{2.7}$$

and the CKY tensor is written as

$$h = \sum_{\mu=1}^{n} x_\mu dx_\mu \wedge \left(\sum_{k=0}^{n-1}\sigma_k(\hat{x}_\mu)\theta_k\right) + \sum_{i=1}^{N}\xi_i\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2)\omega^{(i)}, \tag{2.8}$$

where the function P_μ is defined by

$$P_\mu(x) = \frac{X_\mu}{(x_\mu)^{m^{(0)}}\prod_{i=1}^{N}(x_\mu^2 - \xi_i^2)^{m_i}U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n}(x_\mu^2 - x_\nu^2) \tag{2.9}$$

with an arbitrary function $X_\mu = X_\mu(x_\mu)$ depending only on x_μ. In these expressions

(a) the coordinates x_μ ($\mu = 1, \cdots, n$) and the parameters ξ_i ($i = 1, \cdots, N$) are the non-constant eigenvalues and the non-zero constant eigenvalues of h, respectively.

(b) the dimension $2m_i$ of the Kähler manifold $M^{(i)}$ is equal to multiplicity of non-zero constant eigenvalue ξ_i, and the dimension $m^{(0)}$ of the Riemann manifold $M^{(0)}$ is equal to multiplicity of zero eigenvalue.

(c) for $m^{(0)} = 1$ the last term in (2.7) can take the special form:

$$\sigma_n g^{(0)} \quad \longrightarrow \quad \sigma_n g^{(0)}_{\text{special}} = \frac{c}{\sigma_n} \left(\sum_{k=0}^{n} \sigma_k \theta_k \right)^2 \tag{2.10}$$

with a constant c. Here σ_k is the k-th elementary symmetric functions of $\{x_1^2, \dots, x_n^2\}$:

$$\prod_{\nu=1}^{n} (t - x_\nu^2) = \sigma_0 t^n - \sigma_1 t^{n-1} + \cdots + (-1)^n \sigma_n. \tag{2.11}$$

We note that the odd-dimensional Kerr-NUT-de Sitter spacetime belongs to the special type with $N = 0$.

(d) the 1-forms θ_k satisfy

$$d\theta_k + 2 \sum_{i=1}^{N} (-1)^{(n-k)} \xi_i^{2n-2k-1} \omega^{(i)} = 0, \qquad k = 0, 1, \dots, n-1+\varepsilon, \tag{2.12}$$

where $\omega^{(i)}$ is a Kähler form on $M^{(i)}$ and $\varepsilon = 0$ for the general type and $\varepsilon = 1$ for the special type.

3 Einstein condition and black hole solutions

For the generalized Kerr-NUT-de Sitter spacetime (M, g) the Einstein condition

$$Ric(g) = \Lambda g \tag{3.1}$$

was explicitly solved in [14]. In appendices A and B we give explicit forms of the spin connection 1-forms and the Riemann curvature 2-forms which were omitted in [14].

The result is as follows: the base metrics $g^{(0)}$ and $g^{(i)}$ are Einstein and the function X_μ in (2.9) takes the form

$$X_\mu(x_\mu) = x_\mu \left(d_\mu + \int \chi(x_\mu)\, {x_\mu}^{m^{(0)}-2} \prod_{i=1}^{N} (x_\mu^2 - \xi_i^2)^{m_i} dx_\mu \right), \tag{3.2}$$

where

$$\chi(x) = \sum_{i=-\varepsilon}^{n} \alpha_i x^{2i}, \quad \alpha_n = -\Lambda, \tag{3.3}$$

(a) general type ($\varepsilon = 0$)

$$\alpha_0 = (-1)^{n-1}\lambda^{(0)}, \tag{3.4}$$

(b) special type ($m^{(0)} = 1$ and $\varepsilon = 1$)

$$\alpha_0 = (-1)^{n-1} 2c \sum_{j=1}^{N} \frac{m_j}{\xi_j^2}, \quad \alpha_{-1} = (-1)^{n-1} 2c. \tag{3.5}$$

Here, α_i and d_μ are constants. The $\lambda^{(0)}$ in (3.4) is a cosmological constant of $g^{(0)}$, and cosmological constants $\lambda^{(i)}$ of $g^{(i)}$ are given by

$$\lambda^{(i)} = (-1)^{n-1}\chi(\xi_i). \tag{3.6}$$

To illustrate the above solutions (3.2), in the following subsections 3.1 and 3.2, we consider a subfamily of the solutions which represents the Kerr-de Sitter black holes with partially equal angular momenta (and with some zero angular momenta).

3.1 Special type ($D = 2n + 2|m| + 1$)

First we consider a particular subset of the special case ($m^{(0)} = 1, \varepsilon = 1$). Let us set the number of base Kähler manifolds $N = n$. We choose n Kähler manifolds $M^{(i)}$ to be the complex projective spaces $\mathbb{CP}^{m_i}$. The base space B is

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(n)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_n}, \tag{3.7}$$

and the fiber will be chosen as a $(2n+1)$-dimensional Kerr-de Sitter spacetime.

We restrict various parameters as follows

$$D = 2n + 2|m| + 1, \qquad |m| = \sum_{i=1}^{n} m_i, \qquad \Lambda = (D-1)\lambda = 2(n + |m|)\lambda. \tag{3.8}$$

The metric of special type with $N = n$ is given by

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu)\theta_k\right]^2 + \frac{c}{\sigma_n}\left[\sum_{k=0}^{n} \sigma_k\theta_k\right]^2 + \sum_{i=1}^{n}\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2) g^{(i)}. \tag{3.9}$$

We choose the functions P_μ and the constant c as follows:

$$P_\mu = \frac{X_\mu(x_\mu)}{x_\mu \prod_{i=1}^{n}(x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \qquad c = -\prod_{i=1}^{n} \xi_i^2, \tag{3.10}$$

$$X_\mu(x_\mu) = x_\mu \left(\tilde{d}_\mu - \left(1 + \lambda x_\mu^2\right) x_\mu^{-2} \prod_{i=1}^{n}(x_\mu^2 - \xi_i^2)^{m_i+1} \right). \tag{3.11}$$

Here $g^{(i)}$ is the Fubini-Study metric on $\mathbb{CP}^{m_i}$ with the cosmological constant $\lambda^{(i)}$. The constants $\tilde{d}_\mu$ correspond to the mass M and the NUT parameters. We set all NUT parameters to be zero:

$$\tilde{d}_\mu = (-1)^{(1/2)(D-1)-1} 2M \delta_{\mu,n}. \tag{3.12}$$

We can show that the above metric (3.9) represents the $D = 2n' + 1$ dimensional Kerr-de Sitter metric [16, 17] with partially equal angular momenta: for $n' = n + |m|$, among n' angular momenta $\{a_I\}$, $m_i + 1$ a's are taken equal to ξ_i ($i = 1, 2, \ldots, n$). Here the non-zero constants ξ_i are assumed to be all different: $\xi_i \neq \xi_j$ ($\forall i \neq \forall j$). See Appendix C for details.

The function $X_\mu(x_\mu)$ (3.11) is indeed the special case of (3.2). The corresponding function $\chi(x)$ (3.3) in this case is given by

$$\begin{aligned} \chi(x) = \sum_{i=-1}^{n} \alpha_i x^{2i} &= -2 \sum_{i=1}^{n} (m_i + 1)(1 + \lambda \xi_i^2) \prod_{\substack{j=1 \\ (j \neq i)}}^{n} (x^2 - \xi_j^2) \\ &\quad + 2 \left(\frac{1}{x^2} - (n + |m|)\lambda \right) \prod_{i=1}^{n} (x^2 - \xi_i^2). \end{aligned} \tag{3.13}$$

Explicit form of some parameters α_i can be read off:

$$\alpha_{-1} = (-1)^n 2 \prod_{i=1}^{n} \xi_i^2, \qquad \alpha_0 = (-1)^n 2 \prod_{i=1}^{n} \xi_i^2 \left(\sum_{j=1}^{n} \frac{m_j}{\xi_j^2} \right), \qquad \alpha_n = -2(n + |m|)\lambda. \tag{3.14}$$

The cosmological constant $\lambda^{(i)}$ of $g^{(i)}$ is given by

$$\lambda^{(i)} = (-1)^{n-1} \chi(\xi_i) = (-1)^n 2(m_i + 1)(1 + \lambda \xi_i^2) \prod_{\substack{j=1 \\ (j \neq i)}}^{n} (\xi_i^2 - \xi_j^2). \tag{3.15}$$

3.2 General type ($D = 2n + 2|m| + m^{(0)}$)

Next we consider a particular subset of the general type ($\varepsilon = 0$). Let the number of base Kähler manifolds be $N = n - 1$, and take $M^{(i)} = \mathbb{CP}^{m_i}$ ($i = 1, 2, \ldots, n-1$). Also we take $M^{(0)}$ to be an $m^{(0)}$-dimensional sphere $S^{m^{(0)}}$. The base space B is

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(n-1)} \times M^{(0)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_{n-1}} \times S^{m^{(0)}}. \quad (3.16)$$

We will choose the fiber as a $2n$-dimensional Kerr-de Sitter spacetime. Hence

$$D = 2n + 2|m| + m^{(0)}, \qquad |m| = \sum_{i=1}^{n-1} m_i, \qquad \Lambda = (D-1)\lambda. \quad (3.17)$$

The metric of general type with $N = n - 1$ is given by

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu)\theta_k \right]^2 + \sum_{i=1}^{n-1} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_i^2) g^{(i)} + \sigma_n g^{(0)}. \quad (3.18)$$

Here $g^{(i)}$ is the Fubini-Study metric on $\mathbb{CP}^{m_i}$ with the cosmological constant $\lambda^{(i)}$ and $g^{(0)}$ is the standard metric on the sphere $S^{m^{(0)}}$ with the cosmological constant $\lambda^{(0)}$. The functions P_μ are chosen as follows:

$$P_\mu = \frac{X_\mu(x_\mu)}{(x_\mu)^{m^{(0)}} \prod_{i=1}^{n-1} (x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \qquad \mu = 1, 2, \ldots, n, \quad (3.19)$$

where

$$X_\mu(x_\mu) = x_\mu \left(\tilde{d}_\mu - (1 + \lambda x_\mu^2) x_\mu^{m^{(0)}-1} \prod_{i=1}^{n-1} (x_\mu^2 - \xi_i^2)^{m_i+1} \right), \quad (3.20)$$

with zero NUT parameters

$$\tilde{d}_\mu = \begin{cases} (-1)^{(1/2)D-1} 2Mi\delta_{\mu,n} & (D \text{ even}) \\ (-1)^{(1/2)(D-1)-1} 2M\delta_{\mu,n} & (D \text{ odd}). \end{cases} \quad (3.21)$$

This metric (3.18) represents the general Kerr-de Sitter metric [16, 17] with partially equal angular momenta and with some zero angular momenta. See Appendix C.1 and D for details.

In this case, the function (3.3) is given by

$$\chi(x) = \sum_{i=0}^{n} \alpha_i x^{2i} = -2(1+\lambda x^2)\sum_{i=1}^{n-1}(m_i+1)\xi_i^2 \prod_{\substack{j=1 \\ (j\neq i)}}^{n-1}(x^2-\xi_j^2) \\ + \Big(2-(D-1)(1+\lambda x^2)\Big)\prod_{j=1}^{n-1}(x^2-\xi_j^2). \tag{3.22}$$

Here

$$\alpha_0 = (-1)^{n-1}\lambda^{(0)}, \qquad \lambda^{(0)} = -(m^{(0)}-1)\prod_{i=1}^{n-1}\xi_i^2, \tag{3.23}$$

$$\lambda^{(i)} = (-1)^{n-1}\chi(\xi_i) = (-1)^n 2(m_i+1)(1+\lambda\xi_i^2)\xi_i^2 \prod_{\substack{j=1 \\ (j\neq i)}}^{n-1}(\xi_i^2-\xi_j^2). \tag{3.24}$$

Note that the standard metric $d\Omega^2_{(m^{(0)})}$ on $S^{m^{(0)}}$ with unit radius has the cosmological constant $(m^{(0)}-1)$. Hence,

$$g^{(0)} = -\left(\prod_{i=1}^{n-1}\xi_i^{-2}\right) d\Omega^2_{(m^{(0)})}. \tag{3.25}$$

4 Separability of gravitational perturbation

In this section we study a linear perturbation $g_{AB} \to g_{AB} + h_{AB}$ of the generalized Kerr-NUT-de Sitter metric $g = (g_{AB})$. We assume that the background metric satisfies the Einstein condition. Under the traceless and transverse conditions

$$g^{AB}h_{AB} = 0, \quad \nabla^A h_{AB} = 0, \tag{4.1}$$

the linearized Einstein equation is given by

$$\Delta_L h_{AB} = 2\Lambda h_{AB}, \tag{4.2}$$

where the Lichnerowicz operator Δ_L is defined by

$$\Delta_L h_{AB} = -\nabla^C\nabla_C h_{AB} - 2R_{ACBD}h^{CD} + 2\Lambda h_{AB}. \tag{4.3}$$

Using the orthonormal frame $\{e_A\}$, we have

$$\nabla_A h_{BC} = e_A(h_{BC}) - h_{DC}\omega_{DB}(e_A) - h_{BD}\omega_{DC}(e_A), \tag{4.4}$$

$$\begin{aligned}\nabla_C\nabla_C h_{AB} = & \, e_C(\nabla_C h_{AB}) - (\nabla_D h_{AB})\omega_{DC}(e_C) \\ & - (\nabla_C h_{DB})\omega_{DA}(e_C) - (\nabla_C h_{AD})\omega_{DB}(e_C).\end{aligned} \tag{4.5}$$

Before describing details of perturbation, we summarize the index labeling. Recall that the spacetime has a bundle structure. We use

(i) μ or $n+\mu$ ($\mu = 1, 2, \ldots, n$) for the Kerr-NUT-de Sitter fiber spacetime,

(ii) $(\hat{\alpha}, i) = (\alpha, i)$ or $(m_i + \alpha, i)$ ($\alpha = 1, \ldots, m_i$) for the i-th base Kähler manifold $M^{(i)}$,

(iii-1) a ($a = 1, \ldots, m^{(0)}$) for the general type base Riemann manifold $M^{(0)}$,

(iii-2) $2n+1$ for the special type base Riemann manifold $M^{(0)}$.

The components h_{AB} can be classified into scalar, vector and tensor components according to the coordinate transformations of the base manifolds $M^{(0)}$ and $M^{(i)}$. Here we consider the perturbation of the base manifolds $B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(N)} \times M^{(0)}$. We call such perturbation a tensor perturbation according to [23]. We will show later that the tensor components are decoupled from other components (see below equation (4.16)). Hence, one can consistently require that

(a) no scalar component

$$\begin{aligned} h_{\mu\nu} = h_{\mu,n+\nu} = h_{n+\mu,n+\nu} = 0, \\ h_{\mu,2n+1} = h_{n+\mu,2n+1} = h_{2n+1,2n+1} = 0, \end{aligned} \tag{4.6}$$

(b) no vector component

$$\begin{aligned} h_{\mu,(\hat{\alpha},i)} = h_{n+\mu,(\hat{\alpha},i)} = h_{2n+1,(\hat{\alpha},i)} = 0, \\ h_{\mu a} = h_{n+\mu,a} = 0. \end{aligned} \tag{4.7}$$

These conditions (a) and (b) mean that we do not perturb the fiber metric and keep the bundle structure.

For simplicity, we further impose the following conditions on the tensor components:

$$\sum_{\hat{\alpha}=1}^{2m_i} h_{(\hat{\alpha},i),(\hat{\alpha},i)} = \sum_{\alpha=1}^{m_i} (h_{(\alpha,i),(\alpha,i)} + h_{(m_i+\alpha,i),(m_i+\alpha,i)}) = 0 \quad \text{for each } i, \tag{4.8}$$

$$\sum_{a=1}^{m^{(0)}} h_{aa} = 0 \tag{4.9}$$

and

$$h_{(\hat{\alpha},i),(\hat{\beta},j)} = 0 \quad \text{for} \quad i \neq j, \quad h_{(\hat{\alpha},i),a} = 0. \tag{4.10}$$

Now, the traceless condition is automatically satisfied and the transverse condition reduces to

$$\begin{aligned}\mathcal{D}^{(i)}_{\hat{\alpha}} h_{(\hat{\alpha},i),(\hat{\beta},i)} &:= \bar{e}^{(i)}_{\hat{\alpha}}(h_{(\hat{\alpha},i),(\hat{\beta},i)}) - h_{(\hat{\gamma},i),(\hat{\beta},i)} \tilde{\omega}_{(\hat{\gamma},i),(\hat{\alpha},i)}(\tilde{e}^{(i)}_{\hat{\alpha}}) - h_{(\hat{\alpha},i),(\hat{\gamma},i)} \tilde{\omega}_{(\hat{\gamma},i),(\hat{\beta},i)}(\tilde{e}^{(i)}_{\hat{\alpha}}) \\ &= 0, \end{aligned}\tag{4.11}$$

$$\begin{aligned} D^{(0)}_a h_{ab} &:= \tilde{e}_a(h_{ab}) - h_{cb}\tilde{\omega}_{ca}(\tilde{e}_a) - h_{ac}\tilde{\omega}_{cb}(\tilde{e}_a) \\ &= 0, \end{aligned}\tag{4.12}$$

where $\mathcal{D}^{(i)}$ is the gauge-covariant derivative on the Kähler-Einstein manifold $M^{(i)}$ and $D^{(0)}$ the covariant derivative on the Einstein manifold $M^{(0)}$. It should be noticed that $\mathcal{D}^{(i)}$ includes the 1-form $A^{(i)}$ given by the Kähler form $\omega^{(i)} = dA^{(i)}$(see (A.4) and (B.3)).

4.1 General type

Now, we show that the equation (4.2) allows a separation of variables for the tensor components

$$h_{(\hat{\alpha},i),(\hat{\beta},i)} = \left(\prod_{\mu=1}^{n} A^{(i)}_{\mu}(x_\mu) \prod_{k=0}^{n-1} e^{iN_k\psi_k} \right) H^{(i)}_{\hat{\alpha}\hat{\beta}}(y^{(i)}_I) \prod_{\substack{j=1 \\ (j\neq i)}}^{N} K^{(j)}(y^{(j)}_J) K^{(0)}(z_M), \tag{4.13}$$

$$h_{ab} = \left(\prod_{\mu=1}^{n} B_{\mu}(x_\mu) \prod_{k=0}^{n-1} e^{iN_k\psi_k} \right) \prod_{i=1}^{N} K^{(i)}(y^{(i)}_I) H^{(0)}_{ab}(z_M), \tag{4.14}$$

where $H^{(i)}_{\hat{\alpha}\hat{\beta}}(y^{(i)}_I)$ and $H^{(0)}_{ab}(z_M)$ are tensor components on $M^{(i)}$ and $M^{(0)}$ respectively. $K^{(i)}(y^{(i)}_I)$ and $K^{(0)}(z_M)$ are scalar functions on $M^{(i)}$ and $M^{(0)}$. Also, $\{y^{(i)}_I; I = 1, \cdots, 2m_i\}$

and $\{z_M; M = 1, \cdots, m^{(0)}\}$ represent the local coordinates on these spaces. Scalar components of (4.2) are trivially satisfied by the condition (4.6). We require the equation

$$\mathcal{D}^{(i)}_{\beta} H^{(i)}_{m_i+\beta,\hat{\alpha}} - \mathcal{D}^{(i)}_{m_i+\beta} H^{(i)}_{\beta,\hat{\alpha}} = 0 \tag{4.15}$$

together with the transverse conditions:

$$\mathcal{D}^{(i)}_{\hat{\alpha}} H^{(i)}_{\hat{\alpha}\hat{\beta}} = 0, \qquad D^{(0)}_a H^{(0)}_{ab} = 0. \tag{4.16}$$

This is a consequence of the vector component (E.1). Thus the tensor components are decoupled from the scalar and vector components. It should be noticed that from (A.4) the derivative $\bar{e}^{(i)}_{\hat{\alpha}}$ in the gauge-covariant derivative is given by

$$\bar{e}^{(i)}_{\hat{\alpha}} := \tilde{e}^{(i)}_{\hat{\alpha}} + i n_i A^{(i)}_{\hat{\alpha}}, \tag{4.17}$$

where

$$n_i = 2\sum_{k=0}^{n-1}(-1)^{n+k}\xi_i^{2(n-k)-1}N_k. \tag{4.18}$$

If we choose the Killing coordinates ψ_i suitably, then the charge n_i is proportional to an integer, which is interpreted as the first Chern number of the line bundle over the Kähler-Einstein manifold $M^{(i)}$.

Let us evaluate the tensor components. The non-zero tensor components of the Lichnerowicz operator $\Delta_L h_{AB}$ are given in Appendix E.1. The $\Box^{(F)}$-part was already studied in the separability of the Klein-Gordon equation on the Kerr-NUT-de Sitter background [33]. Using the vector fields (A.3) and the identity

$$\frac{1}{\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2)} = (-1)^{n+1}\sum_{\mu=1}^{n}\frac{1}{U_\mu(x_\mu^2 - \xi_i^2)}, \tag{4.19}$$

we find that the equation (4.2) takes the form

$$\sum_{\mu=1}^{n}\frac{1}{U_\mu}G^{(i)}_{\hat{\alpha}\hat{\beta}}(x_\mu; y^{(i)}_I) = 0, \qquad \sum_{\mu=1}^{n}\frac{1}{U_\mu}G_{ab}(x_\mu; z_M) = 0, \tag{4.20}$$

where $G^{(i)}_{\hat{\alpha}\hat{\beta}}$ and G_{ab} include the single coordinate x_μ. The explicit forms are given by

$$\begin{aligned}
G^{(i)}_{\alpha\beta} &= L^{(i)}(x_\mu)H^{(i)}_{\alpha\beta} + iM^{(i)}(x_\mu)(H^{(i)}_{m_i+\alpha,\beta} + H^{(i)}_{\alpha,m_i+\beta}) + N^{(i)}(x_\mu)H^{(i)}_{m_i+\alpha,m_i+\beta}, \\
G^{(i)}_{\alpha,m_i+\beta} &= L^{(i)}(x_\mu)H^{(i)}_{\alpha,m_i+\beta} + iM^{(i)}(x_\mu)(H^{(i)}_{m_i+\alpha,m_i+\beta} - H^{(i)}_{\alpha,\beta}) - N^{(i)}(x_\mu)H^{(i)}_{m_i+\alpha,\beta}, \\
G^{(i)}_{m_i+\alpha,m_i+\beta} &= L^{(i)}(x_\mu)H^{(i)}_{m_i+\alpha,m_i+\beta} - iM^{(i)}(x_\mu)(H^{(i)}_{\alpha,m_i+\beta} + H^{(i)}_{m_i+\alpha,\beta}) + N^{(i)}(x_\mu)H^{(i)}_{\alpha,\beta},
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
L^{(i)} &= -\frac{1}{A^{(i)}_\mu}\frac{d}{dx_\mu}\tilde{X}_\mu\frac{d}{dx_\mu}A^{(i)}_\mu + \frac{1}{\tilde{X}_\mu}\sum_{k,\ell=0}^{n-1}(-1)^{k+\ell}N_kN_\ell x_\mu^{2(2n-k-\ell-2)} \\
&\quad + \sum_{j=1(j\neq i)}^{N}\frac{(-1)^{n+1}}{x_\mu^2-\xi_j^2}\frac{\Box^{(j)}K^{(j)}}{K^{(j)}} - 2\sum_{j=1}^{N}\frac{m_jx_\mu\tilde{X}_\mu}{(x_\mu^2-\xi_j^2)A^{(i)}_\mu}\frac{d}{dx_\mu}A^{(i)}_\mu \\
&\quad + \frac{(-1)^{n+1}}{x_\mu^2}\frac{\Box^{(0)}K^{(0)}}{K^{(0)}} - \frac{m^{(0)}\tilde{X}_\mu}{x_\mu A^{(i)}_\mu}\frac{d}{dx_\mu}A^{(i)}_\mu \\
&\quad + \frac{4\xi_i^2\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)^2} + \frac{(-1)^{n+1}}{x_\mu^2-\xi_i^2}(\Delta_L^{(i)}-2\lambda^{(i)}),
\end{aligned} \tag{4.22}$$

$$M^{(i)} = \frac{2\xi_i}{x_\mu^2-\xi_i^2}\sum_{k=0}^{n-1}(-1)^k x_\mu^{2(n-k-1)}N_k, \tag{4.23}$$

$$N^{(i)} = 4\sum_{j=1}^{N}\frac{\xi_i\xi_j\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)}$$

and

$$G_{ab} = R(x_\mu)H^{(0)}_{ab}, \tag{4.24}$$

where

$$\begin{aligned}
R &= -\frac{1}{B_\mu}\frac{d}{dx_\mu}\tilde{X}_\mu\frac{d}{dx_\mu}B_\mu + \frac{1}{\tilde{X}_\mu}\sum_{k,\ell=0}^{n-1}(-1)^{k+\ell}N_kN_\ell x_\mu^{2(2n-k-\ell-2)} \\
&\quad + \sum_{i=1}^{N}\frac{(-1)^{n+1}}{x_\mu^2-\xi_i^2}\frac{\Box^{(i)}K^{(i)}}{K^{(i)}} - 2\sum_{i=1}^{N}\frac{m_ix_\mu\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)B_\mu}\frac{d}{dx_\mu}B_\mu \\
&\quad - \frac{m^{(0)}\tilde{X}_\mu}{x_\mu B_\mu}\frac{d}{dx_\mu}B_\mu + \frac{(-1)^{n+1}}{x_\mu^2}(\Delta_L^{(0)}-2\lambda^{(0)}).
\end{aligned} \tag{4.25}$$

In these expressions the function $\tilde{X}_\mu$ is defined by

$$\tilde{X}_\mu = \frac{1}{(x_\mu)^{m^{(0)}-1}\prod_{i=1}^{N}(x_\mu^2-\xi_i^2)^{m_i}}\left(d_\mu + \int \chi(x_\mu){x_\mu}^{m^{(0)}-2}\prod_{i=1}^{N}(x_\mu^2-\xi_i^2)^{m_i}dx_\mu\right). \tag{4.26}$$

The operator $\Delta_L^{(i)}$ is the gauge-covariant Lichnerowicz operator on $M^{(i)}$:

$$\Delta_L^{(i)}H^{(i)}_{\hat{\alpha}\hat{\beta}} = -\sum_{\hat{\gamma}}\mathcal{D}^{(i)}_{\hat{\gamma}}\mathcal{D}^{(i)}_{\hat{\gamma}}H^{(i)}_{\hat{\alpha}\hat{\beta}} - 2\sum_{\hat{\gamma},\hat{\delta}}\tilde{R}^{(i)}_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}H^{(i)}_{\hat{\gamma}\hat{\delta}} + 2\lambda^{(i)}H^{(i)}_{\hat{\alpha}\hat{\beta}} \tag{4.27}$$

and $\Box^{(i)}$ is the gauged scalar Laplacian on $M^{(i)}$:

$$\Box^{(i)}K^{(i)} = -\sum_{\hat{\alpha}}\mathcal{D}^{(i)}_{\hat{\alpha}}\bar{e}^{(i)}_{\hat{\alpha}}K^{(i)}. \tag{4.28}$$

The Lichnerowicz operator $\Delta_L^{(0)}$ on $M^{(0)}$ is defined by

$$\Delta_L^{(0)}H^{(0)}_{ab} = -\sum_{d}D^{(0)}_{d}D^{(0)}_{d}H^{(0)}_{ab} - 2\sum_{de}\tilde{R}^{(i)}_{adbe}H^{(0)}_{de} + 2\lambda^{(0)}H^{(0)}_{ab} \tag{4.29}$$

and $\Box^{(0)}$ is the scalar Laplacian on $M^{(0)}$:

$$\Box^{(0)}K^{(0)} = -\sum_{a}D^{(0)}_{a}\tilde{e}_aK^{(0)}. \tag{4.30}$$

The cosmological constants $\lambda^{(i)}$ and $\lambda^{(0)}$ are given by (3.6) and (3.4).

The solutions of (4.20) are

$$G^{(i)}_{\hat{\alpha}\hat{\beta}} = \sum_{k=0}^{n-2}b^{(i)}_k x_\mu^{2k}H^{(i)}_{\hat{\alpha}\hat{\beta}}, \quad G_{ab} = \sum_{k=0}^{n-2}b_k x_\mu^{2k}H^{(0)}_{ab}, \tag{4.31}$$

where $b^{(i)}_k$ and b_k are arbitrary constants. This is shown by employing the identity

$$\sum_{\mu=1}^{n}\frac{x_\mu^{2k}}{U_\mu} = 0 \qquad \text{for} \quad k = 0, \cdots, n-2, \tag{4.32}$$

which also played crucial roles in [7, 33, 34].

Now we further simplify the perturbation equation with help of the Kähler condition. We can take the tensor $H^{(i)}_{\hat{\alpha}\hat{\beta}}$

- hermitian : $H^{(i)}_{\alpha\beta} = H^{(i)}_{m_i+\alpha,m_i+\beta}, \quad H^{(i)}_{\alpha,m_i+\beta} = -H^{(i)}_{m_i+\alpha,\beta}$ (4.33)

or

- anti-hermitian : $\quad H^{(i)}_{\alpha\beta} = -H^{(i)}_{m_i+\alpha, m_i+\beta}, \; H^{(i)}_{\alpha, m_i+\beta} = H^{(i)}_{m_i+\alpha,\beta}.$ (4.34)

This means that $H^{(i)}_{\hat\alpha\hat\beta}$ is an eigenfunction of the linear map $\mathcal{J}^{(i)}$ [23]

$$(\mathcal{J}^{(i)} H^{(i)})_{\hat\alpha\hat\beta} = J^{(i)\hat\gamma}_{\hat\alpha} J^{(i)\hat\delta}_{\hat\beta} H^{(i)}_{\hat\gamma\hat\delta}, \tag{4.35}$$

where $J^{(i)}$ is given by (A.2). Indeed, the map has the eigenvalues ± 1 and their eigenfunctions are hermitian or anti-hermitian, respectively. Note that the condition (4.15) is now a consequence of the transverse condition (4.16). Furthermore the Kähler condition (A.12) leads to the commutativity $[\Delta^{(i)}_L, \mathcal{J}^{(i)}] = 0$ on the rank-2 tensor space. Thus one can choose the simultaneous eigenfunction $H^{(i)}_{\hat\alpha\hat\beta}$ of the operators $\Delta^{(i)}_L$ and $\mathcal{J}^{(i)}$. It is also assumed that $K^{(i)}$, $K^{(0)}$ and $H^{(0)}_{ab}$ are eigenfunctions of $\Box^{(i)}$, $\Box^{(0)}$ and $\Delta^{(0)}_L$, and $\{H^{(i)}_{\hat\alpha\hat\beta}, H^{(0)}_{ab}\}$ satisfies the traceless and transverse conditions.

(a) hermitian: Combining (4.21) (4.24) with (4.31) we have

$$L^{(i)} + N^{(i)} = \sum_{k=0}^{n-2} b^{(i)}_k x^{2k}_\mu, \quad R = \sum_{k=0}^{n-2} b_k x^{2k}_\mu. \tag{4.36}$$

(b) anti-hermitian : We have

$$\left(L^{(i)} - N^{(i)} - \sum_{k=0}^{n-2} b^{(i)}_k x^{2k}_\mu\right) H^{(i)}_{\alpha\beta} + 2iM^{(i)} H^{(i)}_{\alpha, m_i+\beta} = 0, \tag{4.37}$$

$$-2iM^{(i)} H^{(i)}_{\alpha\beta} + \left(L^{(i)} - N^{(i)} - \sum_{k=0}^{n-2} b^{(i)}_k x^{2k}_\mu\right) H^{(i)}_{\alpha, m_i+\beta} = 0. \tag{4.38}$$

In order to allow the non-zero eigenfunction $H^{(i)}_{\hat\alpha\hat\beta}$, it is necessary to satisfy the condition

$$L^{(i)} - N^{(i)} + 2\epsilon M^{(i)} - \sum_{k=0}^{n-2} b^{(i)}_k x^{2k}_\mu = 0 \tag{4.39}$$

with $\epsilon = \pm 1$. We also obtain the second equation in (4.36).

Thus we have demonstrated that the equation (4.2) in the generalized Kerr-NUT-de Sitter background allows a separation of variables (4.13) (4.14) when the functions $A_\mu^{(i)}$ and B_μ satisfy the ordinary second order differential equations:

$$-\frac{d}{dx_\mu}\tilde{X}_\mu\frac{d}{dx_\mu}A_\mu^{(i)} - \left(2\sum_{j=1}^{N}\frac{m_j x_\mu \tilde{X}_\mu}{x_\mu^2-\xi_j^2} + \frac{m^{(0)}\tilde{X}_\mu}{x_\mu}\right)\frac{d}{dx_\mu}A_\mu^{(i)} + V_\mu^{(i)}A_\mu^{(i)} = 0, \tag{4.40}$$

$$\begin{aligned} V_\mu^{(i)} = {} & \frac{1}{\tilde{X}_\mu}\sum_{k,\ell=0}^{n-1}(-1)^{k+\ell}N_k N_\ell x_\mu^{2(2n-k-\ell-2)} + \sum_{j=1(j\neq i)}^{N}\frac{(-1)^{n+1}E_s^{(j)}}{x_\mu^2-\xi_j^2} + (-1)^{n+1}\frac{E_s^{(0)}}{x_\mu^2} \\ & + \frac{4\xi_i^2\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)^2} + \frac{(-1)^{n+1}(E_t^{(i)}-2\lambda^{(i)})}{x_\mu^2-\xi_i^2} - \sum_{k=0}^{n-2}b_k^{(i)}x_\mu^{2k} \\ & + 4\sigma\sum_{j=1}^{N}\frac{\xi_i\xi_j\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)} + \frac{\epsilon(1-\sigma)2\xi_i}{x_\mu^2-\xi_i^2}\sum_{k=0}^{n-1}(-1)^k x_\mu^{2(n-k-1)}N_k \end{aligned} \tag{4.41}$$

and

$$-\frac{d}{dx_\mu}\tilde{X}_\mu\frac{d}{dx_\mu}B_\mu - \left(2\sum_{j=1}^{N}\frac{m_j x_\mu \tilde{X}_\mu}{x_\mu^2-\xi_j^2} + \frac{m^{(0)}\tilde{X}_\mu}{x_\mu}\right)\frac{d}{dx_\mu}B_\mu + V_\mu^{(0)}B_\mu = 0, \tag{4.42}$$

$$\begin{aligned} V_\mu^{(0)} = {} & \frac{1}{\tilde{X}_\mu}\sum_{k,\ell=0}^{n-1}(-1)^{k+\ell}N_k N_\ell x_\mu^{2(2n-k-\ell-2)} + \sum_{j=1}^{N}\frac{(-1)^{n+1}E_s^{(j)}}{x_\mu^2-\xi_j^2} \\ & + \frac{(-1)^{n+1}(E_t^{(0)}-2\lambda^{(0)})}{x_\mu^2} - \sum_{k=0}^{n-2}b_k x_\mu^{2k}, \end{aligned}$$

where according to hermitian and anti-hermitian the symbol σ takes the values ± 1, and $E_s^{(i)}, E_t^{(i)}, E_s^{(0)}$ and $E_t^{(0)}$ represent the eigenvalues of $\Box^{(i)}, \Delta_L^{(i)}, \Box^{(0)}$ and $\Delta_L^{(0)}$, respectively.

4.2 Special type

The procedure is completely parallel to the case of general type. The equation (4.2) allows a separation of variables for the tensor components

$$h_{(\hat{\alpha},i),(\hat{\beta},i)} = \left(\prod_{\mu=1}^{n}\hat{A}_\mu^{(i)}(x_\mu)\prod_{k=0}^{n}e^{iN_k\psi_k}\right)H_{\hat{\alpha}\hat{\beta}}^{(i)}(y_I^{(i)})\prod_{\substack{j=1\\(j\neq i)}}^{N}K^{(j)}(y_J^{(j)}), \tag{4.43}$$

where $H^{(i)}_{\hat{\alpha}\hat{\beta}}(y^{(i)}_I)$ is the simultaneous eigenfunction of the operators $\Delta^{(i)}_L$ and $\mathcal{J}^{(i)}$ satisfying the traceless and transverse conditions, and $K^{(i)}$ the eigenfunction of $\Box^{(i)}$. The functions $\hat{A}^{(i)}_\mu$ satisfy the ordinary second order differential equations:

$$-\frac{d}{dx_\mu}\tilde{X}_\mu \frac{d}{dx_\mu}\hat{A}^{(i)}_\mu - \left(2\sum_{j=1}^{N}\frac{m_j x_\mu \tilde{X}_\mu}{x_\mu^2-\xi_j^2} + \frac{\tilde{X}_\mu}{x_\mu}\right)\frac{d}{dx_\mu}\hat{A}^{(i)}_\mu + \hat{V}^{(i)}_\mu \hat{A}^{(i)}_\mu = 0, \tag{4.44}$$

$$\begin{aligned}
\hat{V}^{(i)}_\mu =& \frac{1}{\tilde{X}_\mu}\sum_{k,\ell=0}^{n}(-1)^{k+\ell}N_k N_\ell x_\mu^{2(2n-k-\ell-2)} + \frac{(-1)^{n+1}N_n^2}{c x_\mu^2} + \sum_{j=1 (j\neq i)}^{N}\frac{(-1)^{n+1}E_s^{(j)}}{x_\mu^2-\xi_j^2} \\
&+ \frac{4c}{\xi_i^2}\frac{(-1)^{n+1}}{x_\mu^2} + \frac{4\xi_i^2\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)^2} + \frac{(-1)^{n+1}(E_t^{(i)}-2\lambda^{(i)})}{x_\mu^2-\xi_i^2} - \sum_{k=0}^{n-2}\hat{b}^{(i)}_k x_\mu^{2k} \\
&+ 4\sigma\sum_{j=1}^{N}\frac{\xi_i\xi_j\tilde{X}_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)} + \frac{\epsilon(1-\sigma)2\xi_i}{x_\mu^2-\xi_i^2}\sum_{k=0}^{n}(-1)^k x_\mu^{2(n-k-1)}N_k \\
&+ 4\sigma\sum_{j=1}^{N}\frac{(-1)^{n+1}c}{\xi_i\xi_j x_\mu^2} - \frac{(-1)^{n+1}2\epsilon(1-\sigma)N_n}{\xi_i x_\mu^2},
\end{aligned} \tag{4.45}$$

where $\hat{b}^{(i)}_k$ are arbitrary constants and

$$\tilde{X}_\mu = \prod_{i=1}^{N}(x_\mu^2-\xi_i^2)^{-m_i}\left(d_\mu + \int \chi(x_\mu)x_\mu^{-1}\prod_{i=1}^{N}(x_\mu^2-\xi_i^2)^{m_i}dx_\mu\right). \tag{4.46}$$

5 Summary and discussion

In this paper we have studied the separability of the gravitational perturbations in the generalized Kerr-NUT-de Sitter spacetimes. We found that tensor type perturbations admit the separation of variables and the equations of motion reduce to a set of ordinary second order differential equations. It seems to be sure that the separability is deeply related to the existence of the conformal Killing-Yano tensor like the cases of geodesic equation, Klein-Gordon equation and Dirac equation. However, the geometrical origin still remains veiled. It is important to clarify why the separability works well, and also important to study whether the symmetry connected with CKY tensor [35] enables the separation for more general perturbations.

Our results can be used for the study of the stability of higher-dimensional Kerr-de Sitter black holes with partially equal angular momenta. The expressions for the metrics given in [16, 17] are rather complicated and hence they are not so convenient for the perturbations. We found the explicit coordinate transformations from the black holes to the generalized Kerr-NUT-de Sitter spacetimes. Thus our formulation applies to the Kerr-de Sitter black holes with such angular momenta[1]. In order to investigate the problem of the stability we must specify the several quantities in the equations (4.40), (4.42) and (4.44). First we need to know the eigenvalues of the (gauged) scalar Laplacian and (gauge-covariant) Lichnerowicz operator on Kähler-Einstein manifolds and Einstein manifolds. Fortunately, in the case of the Kerr-de Sitter black holes these Einstein manifolds are complex projective spaces and standard spheres, on which the eigenvalues are well known (see for example [36, 37, 38, 39, 40, 41, 42, 23, 43, 44]). Next, we must determine the ranges of the Killing coordinates. If we choose them suitably, then the charge given by (4.18) is proportional to an integer, which is interpreted as the first Chern number of the line bundle over the Kähler-Einstein manifold. The equations (4.40), (4.42) and (4.44) are Fuchs type differential equations, and the boundary condition to the solutions will be given according to the analysis in [23, 31]. We hope to report our stability analysis of the higher-dimensional Kerr-de Sitter black holes in a separated paper.

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[1] Unfortunately, our formulation cannot apply to the non-degenerate angular momenta. This is a future problem.

A Spin connections and curvature 2-forms for the general type

In this appendix, we give explicit forms of the spin connections and curvature 2-forms for the metric (2.7) of the general type.

For the metric (2.7), we introduce the following orthonormal frame $\{e^A\} = \{e^\mu, e^{n+\mu}, e^a, e^{\hat{\alpha}}_{(i)}\}$:

$$e^\mu = \frac{dx_\mu}{\sqrt{P_\mu}}, \qquad e^{n+\mu} = \sqrt{P_\mu}\left(\sum_{k=0}^{n-1}\sigma_k(\tilde{x}_\mu)\theta_k\right),$$
$$e^a = \sqrt{\sigma_n}\tilde{e}^a, \qquad e^{\hat{\alpha}}_{(i)} = \left(\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2)\right)^{1/2}\tilde{e}^{\hat{\alpha}}_{(i)}. \tag{A.1}$$

Here, $\{\tilde{e}^a\}_{a=1,2,\ldots,m^{(0)}}$ is an orthonormal frame of a Riemann manifold $(M^{(0)}, g^{(0)})$, and $\{\tilde{e}^{\hat{\alpha}}_{(i)}\} = \{\tilde{e}^\alpha_{(i)}, \tilde{e}^{m_i+\alpha}_{(i)}\}_{\alpha=1,2,\ldots,m_i}$ are orthonormal frames of Kähler manifolds $(M^{(i)}, g^{(i)}, J^{(i)}, \omega^{(i)})$ $(i = 1, \cdots, N)$ such that the Kähler structure is of the form

$$\bullet\ \text{metric}: \qquad g^{(i)} = \sum_{\alpha=1}^{m_i}(\tilde{e}^\alpha_{(i)} \otimes \tilde{e}^\alpha_{(i)} + \tilde{e}^{m_i+\alpha}_{(i)} \otimes \tilde{e}^{m_i+\alpha}_{(i)}),$$
$$\bullet\ \text{complex structure}: \qquad J^{(i)}(\tilde{e}^\alpha_{(i)}) = -\tilde{e}^{m_i+\alpha}_{(i)}, \quad J^{(i)}(\tilde{e}^{m_i+\alpha}_{(i)}) = \tilde{e}^\alpha_{(i)}, \tag{A.2}$$
$$\bullet\ \text{Kähler form}: \qquad \omega^{(i)} = \sum_{\alpha=1}^{m_i}\tilde{e}^\alpha_{(i)} \wedge \tilde{e}^{m_i+\alpha}_{(i)}.$$

The dual vector fields defined by $e^A(e_B) = \delta^A_B$ are written as

$$e_\mu = \sqrt{P_\mu}\frac{\partial}{\partial x_\mu},$$
$$e_{n+\mu} = \frac{1}{U_\mu\sqrt{P_\mu}}\sum_{k=0}^{n-1}(-1)^k x_\mu^{2(n-k-1)}\frac{\partial}{\partial\psi_k},$$
$$e_a = \frac{1}{\sqrt{\sigma_n}}\tilde{e}_a, \tag{A.3}$$
$$e^{(i)}_{\hat{\alpha}} = \left(\prod_{\mu=1}^{n}(x_\mu^2 - \xi_i^2)\right)^{-1/2}\bar{e}^{(i)}_{\hat{\alpha}},$$

where

$$\bar{e}^{(i)}_{\hat{\alpha}} = \tilde{e}^{(i)}_{\hat{\alpha}} + 2A^{(i)}_{\hat{\alpha}}\sum_{k=0}^{n-1}(-1)^{n+k}\xi_i^{2(n-k)-1}\frac{\partial}{\partial\psi_k}. \tag{A.4}$$

The vector fields $\tilde{e}_a$ and $\tilde{e}^{(i)}_{\hat{\alpha}}$ are the dual vector fields to the 1-forms $\tilde{e}^a$ and $\tilde{e}^{\hat{\alpha}}_{(i)}$, respectively. The $A^{(i)}_{\hat{\alpha}}$ represents the component of the 1-form, $A^{(i)} = A^{(i)}_{\hat{\alpha}} \tilde{e}^{\hat{\alpha}}_{(i)}$.

The connection 1-forms $\omega_{AB} = -\omega_{BA}$, which obey the first structure equation

$$de^A + \omega^A{}_B \wedge e^B = 0, \tag{A.5}$$

are determined as follows:

$$\begin{aligned}
\omega_{\mu\nu} &= \frac{(1-\delta_{\mu\nu})}{x_\mu^2 - x_\nu^2}\left(-x_\nu\sqrt{P_\nu}e^\mu - x_\mu\sqrt{P_\mu}e^\nu\right), \\
\omega_{\mu,n+\nu} &= \delta_{\mu\nu}\left[-\frac{\partial}{\partial x_\mu}(\sqrt{P}_\mu)e^{n+\mu} + \sum_{\rho=1}^{n}\frac{(1-\delta_{\mu\rho})x_\mu\sqrt{P_\rho}}{x_\mu^2 - x_\rho^2}e^{n+\rho}\right] \\
&\quad + \frac{(1-\delta_{\mu\nu})}{x_\mu^2 - x_\nu^2}\left(x_\mu\sqrt{P_\nu}e^{n+\mu} - x_\mu\sqrt{P_\mu}e^{n+\nu}\right), \quad \text{(no sum)}, \\
\omega_{n+\mu,n+\nu} &= \frac{(1-\delta_{\mu\nu})}{x_\mu^2 - x_\nu^2}\left(-x_\nu\sqrt{P_\mu}e^\nu - x_\mu\sqrt{P_\nu}e^\mu\right),
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\omega_{\mu,(\alpha,i)} &= -\frac{x_\mu\sqrt{P_\mu}}{x_\mu^2 - \xi_i^2}e^\alpha_{(i)}, & \omega_{\mu,(m_i+\alpha,i)} &= -\frac{x_\mu\sqrt{P_\mu}}{x_\mu^2 - \xi_i^2}e^{m_i+\alpha}_{(i)}, \\
\omega_{n+\mu,(\alpha,i)} &= \frac{\xi_i\sqrt{P_\mu}}{x_\mu^2 - \xi_i^2}e^{m_i+\alpha}_{(i)}, & \omega_{n+\mu,(m_i+\alpha,i)} &= -\frac{\xi\sqrt{P_\mu}}{x_\mu^2 - \xi_i^2}e^\alpha_{(i)},
\end{aligned} \tag{A.7}$$

$$\omega_{\mu a} = -\frac{\sqrt{P_\mu}}{x_\mu}e^a, \qquad \omega_{n+\mu,a} = 0, \tag{A.8}$$

$$\begin{aligned}
\omega_{(\alpha,i),(\beta,j)} &= \delta_{ij}\tilde{\omega}^{(i)}_{\alpha\beta}, \\
\omega_{(\alpha,i),(m_j+\beta,j)} &= \delta_{ij}\left(\tilde{\omega}^{(i)}_{\alpha,m_i+\beta} - \delta_{\alpha\beta}\sum_{\mu=1}^{n}\frac{\xi_i\sqrt{P_\mu}}{x_\mu^2 - \xi_i^2}e^{n+\mu}\right), \\
\omega_{(m_i+\alpha,i),(m_j+\beta,j)} &= \delta_{ij}\tilde{\omega}^{(i)}_{m_i+\alpha,m_i+\beta},
\end{aligned} \tag{A.9}$$

$$\omega_{(\alpha,i),b} = \omega_{(m_i+\alpha,i),b} = 0, \tag{A.10}$$

$$\omega_{ab} = \tilde{\omega}_{ab}. \tag{A.11}$$

Here $\tilde{\omega}_{ab}$ and $\tilde{\omega}^{(i)}_{\hat{\alpha}\hat{\beta}}$ are connection 1-forms for the metrics of $g^{(0)}$ and $g^{(i)}$, respectively. Since $g^{(i)}$ is a Kähler metric, the 1-forms $\tilde{\omega}^{(i)}_{\hat{\alpha}\hat{\beta}}$ obey the following conditions:

$$\tilde{\omega}^{(i)}_{m_i+\alpha,\beta} = -\tilde{\omega}^{(i)}_{\alpha,m_i+\beta}, \qquad \tilde{\omega}^{(i)}_{m_i+\alpha,m_i+\beta} = \tilde{\omega}^{(i)}_{\alpha\beta}. \tag{A.12}$$

To represent the components of the curvature 2-forms $R^A{}_B$ it is convenient to introduce the following functions:

$$P_T^{[k]}(t) := \sum_{\mu=1}^{n} \frac{P_\mu}{(x_\mu^2 - t)^k}, \quad P_T := P_T^{[0]}(t) = \sum_{\mu=1}^{n} P_\mu. \tag{A.13}$$

We also use the 2-form

$$W^{(i)} := \sum_{\alpha=1}^{m_i} e_{(i)}^{\alpha} \wedge e_{(i)}^{m_i+\alpha}. \tag{A.14}$$

From the second structure equation

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B, \tag{A.15}$$

we find the explicit form of the curvature 2-forms. For $(\mu \neq \nu)$, we have

$$\begin{aligned} R_{\mu\nu} = &-\frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \frac{\partial P_T}{\partial x_\mu} - x_\nu \frac{\partial P_T}{\partial x_\nu} \right) e^\mu \wedge e^\nu \\ &- \frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\nu \frac{\partial P_T}{\partial x_\mu} - x_\mu \frac{\partial P_T}{\partial x_\nu} \right) e^{n+\mu} \wedge e^{n+\nu}, \end{aligned} \tag{A.16}$$

$$\begin{aligned} R_{\mu,n+\mu} = &-\frac{1}{2} \frac{\partial^2 P_T}{\partial x_\mu^2} e^\mu \wedge e^{n+\mu} \\ &+ \sum_{\rho \neq \mu} \frac{1}{x_\mu^2 - x_\rho^2} \left(x_\mu \frac{\partial P_T}{\partial x_\rho} - x_\rho \frac{\partial P_T}{\partial x_\mu} \right) e^\rho \wedge e^{n+\rho} \\ &- \sum_{i=1}^{N} \xi_i \frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu} W^{(i)}, \qquad \text{(no sum)}, \end{aligned} \tag{A.17}$$

$$\begin{aligned} R_{\mu,n+\nu} = &-\frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \frac{\partial P_T}{\partial x_\mu} - x_\nu \frac{\partial P_T}{\partial x_\nu} \right) e^\mu \wedge e^{n+\nu} \\ &+ \frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \frac{\partial P_T}{\partial x_\nu} - x_\nu \frac{\partial P_T}{\partial x_\mu} \right) e^\nu \wedge e^{n+\mu}, \end{aligned} \tag{A.18}$$

$$\begin{aligned} R_{n+\mu,n+\nu} = &-\frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\nu \frac{\partial P_T}{\partial x_\mu} - x_\mu \frac{\partial P_T}{\partial x_\nu} \right) e^\mu \wedge e^\nu \\ &- \frac{1}{2(x_\mu^2 - x_\nu^2)} \left(x_\mu \frac{\partial P_T}{\partial x_\mu} - x_\nu \frac{\partial P_T}{\partial x_\nu} \right) e^{n+\mu} \wedge e^{n+\nu}, \end{aligned} \tag{A.19}$$

$$R_{\mu,(\alpha,i)} = -\left[\left(1 + \frac{1}{2} x_\mu \frac{\partial}{\partial x_\mu} \right) P_T^{[1]}(\xi_i^2) \right] e^\mu \wedge e_{(i)}^{\alpha} - \frac{1}{2} \xi_i \frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu} e^{n+\mu} \wedge e_{(i)}^{m_i+\alpha}, \tag{A.20}$$

$$R_{\mu,(m_i+\alpha,i)} = -\left[\left(1+\frac{1}{2}x_\mu\frac{\partial}{\partial x_\mu}\right)P_T^{[1]}(\xi_i^2)\right]e^\mu\wedge e_{(i)}^{m_i+\alpha} + \frac{1}{2}\xi_i\frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu}e^{n+\mu}\wedge e_{(i)}^\alpha, \quad \text{(A.21)}$$

$$R_{n+\mu,(\alpha,i)} = \frac{1}{2}\xi_i\frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu}e^\mu\wedge e_{(i)}^{m_i+\alpha} - \left[\left(1+\frac{1}{2}x_\mu\frac{\partial}{\partial x_\mu}\right)P_T^{[1]}(\xi_i^2)\right]e^{n+\mu}\wedge e_{(i)}^\alpha, \quad \text{(A.22)}$$

$$R_{n+\mu,(m_i+\alpha,i)} = -\frac{1}{2}\xi_i\frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu}e^\mu\wedge e_{(i)}^\alpha - \left[\left(1+\frac{1}{2}x_\mu\frac{\partial}{\partial x_\mu}\right)P_T^{[1]}(\xi_i^2)\right]e^{n+\mu}\wedge e_{(i)}^{m_i+\alpha}, \quad \text{(A.23)}$$

$$R_{\mu,a} = -\left[\left(1+\frac{1}{2}x_\mu\frac{\partial}{\partial x_\mu}\right)P_T^{[1]}(0)\right]e^\mu\wedge e^a, \quad \text{(A.24)}$$

$$R_{n+\mu,a} = -\left[\left(1+\frac{1}{2}x_\mu\frac{\partial}{\partial x_\mu}\right)P_T^{[1]}(0)\right]e^{n+\mu}\wedge e^a. \quad \text{(A.25)}$$

For $\alpha\neq\beta$ we have

$$\begin{aligned}R_{(\alpha,i),(\beta,i)} &= \tilde{R}_{\alpha\beta}^{(i)} - \left(P_T^{[1]}(\xi_i^2)+\xi_i^2P_T^{[2]}(\xi_i^2)\right)e_{(i)}^\alpha\wedge e_{(i)}^\beta \\ &\quad - \xi_i^2P_T^{[2]}(\xi_i^2)\,e_{(i)}^{m_i+\alpha}\wedge e_{(i)}^{m_i+\beta},\end{aligned} \quad \text{(A.26)}$$

$$\begin{aligned}R_{(\alpha,i),(m_i+\alpha,i)} &= \tilde{R}_{\alpha,m_i+\alpha}^{(i)} - \xi_i\sum_{\mu=1}^n\frac{\partial P_T^{[1]}(\xi_i^2)}{\partial x_\mu}\,e^\mu\wedge e^{n+\mu} \\ &\quad - 2\xi_i^2P_T^{[2]}(\xi_i^2)\,W^{(i)} - 2\sum_{\substack{j=1\\(j\neq i)}}^N\frac{\xi_i\xi_j}{\xi_i^2-\xi_j^2}\left(P_T^{[1]}(\xi_i^2)-P_T^{[1]}(\xi_j^2)\right)W^{(k)} \\ &\quad - \left(P_T^{[1]}(\xi_i^2)+2\xi_i^2P_T^{[2]}(\xi_i^2)\right)e_{(i)}^\alpha\wedge e_{(i)}^{m_i+\alpha}, \qquad \text{(no sum)},\end{aligned} \quad \text{(A.27)}$$

$$\begin{aligned}R_{(\alpha,i),(m_i+\beta,i)} &= \tilde{R}_{\alpha,m_i+\beta}^{(i)} + \xi_i^2P_T^{[2]}(\xi_i^2)\,e_{(i)}^{m_i+\alpha}\wedge e_{(i)}^\beta \\ &\quad - \left(P_T^{[1]}(\xi_i^2)+\xi_i^2P_T^{[2]}(\xi_i^2)\right)e_{(i)}^\alpha\wedge e_{(i)}^{m_i+\beta},\end{aligned} \quad \text{(A.28)}$$

$$\begin{aligned}R_{(m_i+\alpha,i),(m_i+\beta,i)} &= \tilde{R}_{m_i+\alpha,m_i+\beta}^{(i)} - \xi_i^2P_T^{[2]}(\xi_i^2)\,e_{(i)}^\alpha\wedge e_{(i)}^\beta \\ &\quad - \left(P_T^{[1]}(\xi_i^2)+\xi_i^2P_T^{[2]}(\xi_i^2)\right)e_{(i)}^{m_i+\alpha}\wedge e_{(i)}^{m_i+\beta}.\end{aligned} \quad \text{(A.29)}$$

For general α, β and $i\neq j$ we have

$$\begin{aligned}R_{(\alpha,i),(\beta,j)} &= -\frac{1}{\xi_i^2-\xi_j^2}\left(\xi_i^2P_T^{[1]}(\xi_i^2)-\xi_j^2P_T^{[1]}(\xi_j^2)\right)e_{(i)}^\alpha\wedge e_{(j)}^\beta \\ &\quad - \frac{\xi_i\xi_j}{\xi_i^2-\xi_j^2}(P_T^{[1]}(\xi_i^2)-P_T^{[1]}(\xi_j^2))\,e_{(i)}^{m_i+\alpha}\wedge e_{(j)}^{m_i+\beta},\end{aligned} \quad \text{(A.30)}$$

$$
\begin{aligned}
R_{(\alpha,i),(m_i+\alpha,j)} = &-\frac{1}{\xi_i^2-\xi_j^2}\left(\xi_i^2 P_T^{[1]}(\xi_i^2)-\xi_j^2 P_T^{[1]}(\xi_j^2)\right) e^{\alpha}_{(i)}\wedge e^{m_j+\beta}_{(j)} \\
&-\frac{\xi_i\xi_j}{\xi_i^2-\xi_j^2}(P_T^{[1]}(\xi_i^2)-P_T^{[1]}(\xi_j^2))\, e^{m_i+\alpha}_{(i)}\wedge e^{\beta}_{(j)},
\end{aligned} \tag{A.31}
$$

$$
\begin{aligned}
R_{(m_i+\alpha,i),(m_j+\beta,j)} = &-\frac{1}{\xi_i^2-\xi_j^2}\left(\xi_i^2 P_T^{[1]}(\xi_i^2)-\xi_j^2 P_T^{[1]}(\xi_j^2)\right) e^{m_i+\alpha}_{(i)}\wedge e^{m_j+\beta}_{(j)} \\
&-\frac{\xi_i\xi_j}{\xi_i^2-\xi_j^2}(P_T^{[1]}(\xi_i^2)-P_T^{[1]}(\xi_j^2))\, e^{\alpha}_{(i)}\wedge e^{\beta}_{(j)},
\end{aligned} \tag{A.32}
$$

$$
R_{(\alpha,i),b} = -P_T^{[1]}(\xi_i^2) e^{\alpha}_{(i)}\wedge e^b, \tag{A.33}
$$

$$
R_{(m_i+\alpha,i),b} = -P_T^{[1]}(\xi_i^2) e^{m_i+\alpha}_{(i)}\wedge e^b, \tag{A.34}
$$

$$
R_{ab} = \tilde{R}_{ab} - P_T^{[1]}(0)\, e^a\wedge e^b. \tag{A.35}
$$

B Spin connections and curvature 2-forms for the special type

In this appendix, we give explicit forms of the spin connections and the Riemann curvature 2-forms for the metric of the special type: we consider the metric (2.7) with a special metric $g^{(0)}_{\text{special}}$ given by (2.10).

Let us introduce an orthonormal frame $\{\hat{e}^A\} = \{\hat{e}^\mu, \hat{e}^{n+\mu}, \hat{e}^{2n+1}, \hat{e}^{\hat{\alpha}}_{(i)}\}$:

$$
\hat{e}^\mu = e^\mu, \quad \hat{e}^{n+\mu} = e^{n+\mu}, \quad \hat{e}^{2n+1} = \sqrt{S}\sum_{k=0}^{n}\sigma_k\theta_k, \quad \hat{e}^{\hat{\alpha}}_{(i)} = e^{\hat{\alpha}}_{(i)}, \tag{B.1}
$$

where $S = c/\sigma_n$. Then, the dual vector fields are

$$
\begin{aligned}
\hat{e}_\mu &= e_\mu, \quad \hat{e}_{n+\mu} = e_{n+\mu} + \frac{(-1)^n}{U_\mu\sqrt{P_\mu}x_\mu^2}\frac{\partial}{\partial\psi_n}, \\
\hat{e}_{2n+1} &= \left(\frac{S}{c^2}\right)^{1/2}\frac{\partial}{\partial\psi_n}, \quad \hat{e}^{(i)}_{\hat{\alpha}} = \left(\prod_{\mu=1}^{n}(x_\mu^2-\xi_i^2)\right)^{-1/2}\bar{e}^{(i)}_{\hat{\alpha}},
\end{aligned} \tag{B.2}
$$

where

$$
\bar{e}^{(i)}_{\hat{\alpha}} = \tilde{e}^{(i)}_{\hat{\alpha}} + 2A^{(i)}_{\hat{\alpha}}\sum_{k=0}^{n}(-1)^{n+k}\xi_i^{2(n-k)-1}\frac{\partial}{\partial\psi_k}. \tag{B.3}
$$

The connection 1-forms ω_{AB} are given by

$$\begin{aligned}
\hat{\omega}_{\mu\nu} &= \omega_{\mu\nu}, \quad \hat{\omega}_{n+\mu,n+\nu} = \omega_{n+\mu,n+\nu}, \\
\hat{\omega}_{\mu,n+\nu} &= \omega_{\mu,n+\nu} + \delta_{\mu\nu}\frac{\sqrt{S}}{x_\mu}\hat{e}^{2n+1}, \\
\hat{\omega}_{\mu,2n+1} &= \frac{\sqrt{S}}{x_\mu}\hat{e}^{n+\mu} - \frac{\sqrt{P_\mu}}{x_\mu}\hat{e}^{2n+1}, \quad \hat{\omega}_{n+\mu,2n+1} = -\frac{\sqrt{S}}{x_\mu}\hat{e}^{\mu},
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\hat{\omega}_{\mu,(\alpha,i)} &= \omega_{\mu,(\alpha,i)}, \quad \hat{\omega}_{\mu,(m_i+\alpha,i)} = \omega_{\mu,(m_i+\alpha,i)}, \\
\hat{\omega}_{n+\mu,(\alpha,i)} &= \omega_{n+\mu,(\alpha,i)}, \quad \hat{\omega}_{n+\mu,(m_i+\alpha,i)} = \omega_{n+\mu,(m_i+\alpha,i)},
\end{aligned} \tag{B.5}$$

$$\hat{\omega}_{2n+1,(\alpha,i)} = -\frac{\sqrt{S}}{\xi_i}e^{m_i+\alpha}_{(i)}, \qquad \hat{\omega}_{2n+1,(m_i+\alpha,i)} = \frac{\sqrt{S}}{\xi_i}e^{\alpha}_{(i)}, \tag{B.6}$$

$$\begin{aligned}
\hat{\omega}_{(\alpha,i),(\beta,j)} &= \omega_{(\alpha,i),(\beta,j)}, \\
\hat{\omega}_{(\alpha,i),(m_j+\beta,j)} &= \omega_{(\alpha,i),(m_j+\beta,j)} + \delta_{ij}\delta_{\alpha\beta}\frac{\sqrt{S}}{\xi_i}\hat{e}^{2n+1}, \\
\hat{\omega}_{(m_i+\alpha,i),(m_j+\beta,j)} &= \omega_{(m_i+\alpha,i),(m_j+\beta,j)}.
\end{aligned} \tag{B.7}$$

We use a function

$$\hat{P}_T^{[k]}(t) := \sum_{\mu=1}^{n}\frac{P_\mu}{(x_\mu^2 - t)^k} + \frac{S}{(-t)^k} \tag{B.8}$$

with $\hat{P}_T = \hat{P}_T^{[0]}(t)$. Then, the curvature two forms $\hat{R}_{AB}$ are obtained by the replacements $P_T \to \hat{P}_T$ and $P_T^{[k]} \to \hat{P}_T^{[k]}$ in the general type together with

$$\hat{R}_{\mu,2n+1} = -\frac{1}{2x_\mu}\frac{\partial\hat{P}_T}{\partial x_\mu}\hat{e}^{\mu}\wedge\hat{e}^{2n+1}, \quad \hat{R}_{n+\mu,2n+1} = -\frac{1}{2x_\mu}\frac{\partial\hat{P}_T}{\partial x_\mu}\hat{e}^{n+\mu}\wedge\hat{e}^{2n+1}, \tag{B.9}$$

$$\hat{R}_{(\alpha,i),2n+1} = -\hat{P}_T^{[1]}(\xi_i^2)\hat{e}^{\alpha}_{(i)}\wedge\hat{e}^{2n+1}, \quad \hat{R}_{(m_i+\alpha,i),2n+1} = -\hat{P}_T^{[1]}(\xi_i^2)\hat{e}^{m_i+\alpha}_{(i)}\wedge\hat{e}^{2n+1}. \tag{B.10}$$

C $D = 2n' + 1$ Kerr-de Sitter black hole

In this section of Appendix, we will show that the particular subclass of the special case metric (3.9) indeed represent the odd-dimensional Kerr-de Sitter metric with partially equal angular momenta.

$D = 2n' + 1$ Kerr-de Sitter metric [16, 17] is given by

$$g = d\bar{s}^2 + \frac{2M}{U}\left(Wdt + Fdr - \sum_{I=1}^{n'} \frac{a_I \mu_I^2}{1 + \lambda a_I^2} d\phi_I\right)^2, \tag{C.1}$$

where

$$\begin{aligned} d\bar{s}^2 = &-W(1 - \lambda r^2)dt^2 + Fdr^2 + \sum_{I=1}^{n'} \frac{r^2 + a_I^2}{1 + \lambda a_I^2}(d\mu_I^2 + \mu_I^2 d\phi_I^2) \\ &+ \frac{\lambda}{W(1 - \lambda r^2)} \left(\sum_{I=1}^{n'} \frac{(r^2 + a_I^2)\mu_I d\mu_I}{1 + \lambda a_I^2}\right)^2, \end{aligned} \tag{C.2}$$

$$\sum_{I=1}^{n'} \mu_I^2 = 1, \qquad U = \sum_{I=1}^{n'} \frac{\mu_I^2}{r^2 + a_I^2} \prod_{J=1}^{n'} (r^2 + a_J^2). \tag{C.3}$$

$$W = \sum_{I=1}^{n'} \frac{\mu_I^2}{1 + \lambda a_I^2}, \qquad F = \frac{r^2}{1 - \lambda r^2} \sum_{I=1}^{n'} \frac{\mu_I^2}{r^2 + a_I^2}, \tag{C.4}$$

The above Kerr-de Sitter metric has n' angular momenta a_I $(I = 1, 2, \ldots, n')$. In the following part, we require that the angular momenta are partially equal, namely $(m_i + 1)$ of them are chosen to ξ_i $(i = 1, 2, \ldots, n)$:

$$\begin{aligned} a_i &= \xi_1, && i = 1, 2, \ldots, (m_1 + 1), \\ a_{m_1+1+i} &= \xi_2, && i = 1, 2, \ldots, (m_2 + 1), \\ &\cdots \\ a_{m_1+m_2+\cdots+m_{n-1}+n-1+i} &= \xi_n, && i = 1, 2, \ldots, (m_n + 1). \end{aligned} \tag{C.5}$$

Here $\xi_i \neq 0$ $(\forall i)$ and $\xi_i \neq \xi_j$ $(\forall i \neq \forall j)$. Hence,

$$n' = \sum_{i=1}^{n} (m_i + 1) = n + |m|, \qquad |m| := \sum_{i=1}^{n} m_i. \tag{C.6}$$

One will see that to each non-zero constant ξ_i, there corresponds to the complex projective space $\mathbb{CP}^{m_i}$. For this purpose, it is convenient to change the coordinates μ_I into $\{r_i, u_{i,j}\}$:

$$\begin{aligned} \mu_i &= r_1 u_{1,i}, && i = 1, 2, \ldots, (m_1 + 1), \\ \mu_{m_1+1+i} &= r_2 u_{2,i}, && i = 1, 2, \ldots, (m_2 + 1), \\ &\cdots \\ \mu_{m_1+m_2+\cdots+m_{n-1}+n-1+i} &= r_n u_{n,i}, && i = 1, 2, \ldots, (m_n + 1), \end{aligned} \tag{C.7}$$

with constraints

$$\sum_{j=1}^{m_i+1} u_{i,j}^2 = 1, \qquad i = 1, 2, \ldots, n, \qquad \sum_{i=1}^{n} r_i^2 = 1. \tag{C.8}$$

Also the angular variables ϕ_I are renamed $\varphi_{i,j}$:

$$\begin{aligned}
&\varphi_{1,i} := \phi_i, && i = 1, 2, \ldots, (m_1 + 1), \\
&\varphi_{2,i} := \phi_{m_1+1+i}, && i = 1, 2, \ldots, (m_2 + 1), \\
&\cdots \\
&\varphi_{n,i} := \phi_{m_1+m_2+\cdots+m_{n-1}+n-1+i}, && i = 1, 2, \ldots, (m_n + 1).
\end{aligned} \tag{C.9}$$

Then

$$\begin{aligned}
d\bar{s}^2 = &-W(1-\lambda r^2)dt^2 + F dr^2 + \sum_{i=1}^{n} \left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right) \left(dr_i^2 + r_i^2 d\Omega_{i,(2m_i+1)}^2\right) \\
&+ \frac{\lambda}{W(1-\lambda r^2)} \left(\sum_{i=1}^{n} \frac{(r^2+\xi_i^2) r_i dr_i}{1+\lambda\xi_i^2}\right)^2,
\end{aligned} \tag{C.10}$$

where $d\Omega_{i,(2m_i+1)}^2$ is the metric on the sphere S^{2m_i+1} with unit radius:

$$d\Omega_{i,(2m_i+1)}^2 = \sum_{j=1}^{m_i+1} \left(du_{i,j}^2 + u_{i,j}^2 d\varphi_{i,j}^2\right), \qquad \sum_{j=1}^{m_i+1} u_{i,j}^2 = 1. \tag{C.11}$$

In a local coordinate patch where $u_{i,m_i+1} \neq 0$, we take

$$\begin{aligned}
\varphi_{i,j} &=: \psi_i + \chi_{i,j}, \qquad j = 1, 2, \ldots, m_i, \\
\varphi_{i,m_i+1} &=: \psi_i.
\end{aligned} \tag{C.12}$$

We get the Hopf fibration of S^{2m_i+1}, a $U(1)$ bundle over $\mathbb{CP}^{m_i}$:

$$d\Omega_{(2m_i+1)}^2 = (d\psi_i - 2A_i)^2 + d\Sigma_{i,(m_i)}^2, \tag{C.13}$$

where $d\Sigma_{i,(m_i)}^2$ is a real form of the Fubini-Study metric on $\mathbb{CP}^{m_i}$

$$d\Sigma_{i,(m_i)}^2 = \sum_{j=1}^{m_i+1} du_{i,j}^2 + \sum_{j=1}^{m_i} u_{i,j}^2 d\chi_{i,j}^2 - \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} u_{i,j} u_{i,k} d\chi_{i,j} d\chi_{i,k}, \tag{C.14}$$

and A_i is a potential for the corresponding Kähler form $J^{(i)} = dA_i$,

$$A_i = -\frac{1}{2}\sum_{j=1}^{m_i} u_{i,j}^2 d\chi_{i,j}. \tag{C.15}$$

The Fubini-Study metric $d\Sigma^2_{i,(m_i)}$ has the cosmological constant $2(m_i+1)$.

Now the metric (C.1) with partially equal momenta is written as the following form:

$$\begin{aligned} g = &\sum_{i=1}^{n}\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right)^2 dr_i^2 + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+\xi_i^2)r_i dr_i}{1+\lambda\xi_i^2}\right)^2 \\ &+ F dr^2 - W(1-\lambda r^2)dt^2 + \sum_{i=1}^{n}\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right) r_i^2(d\psi_i - 2A_i)^2 \\ &+ \frac{2M}{U}\left(W dt - \sum_{i=1}^{n}\left(\frac{\xi_i r_i^2}{1+\lambda\xi_i^2}\right)(d\psi_i - 2A_i) + F dr\right)^2 \\ &+ \sum_{i=1}^{n}\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right) r_i^2 d\Sigma^2_{i,(m_i)}, \end{aligned} \tag{C.16}$$

with a constraint

$$\sum_{i=1}^{n} r_i^2 = 1. \tag{C.17}$$

The functions in the metric are given by

$$W = \sum_{i=1}^{n}\frac{r_i^2}{1+\lambda\xi_i^2}, \qquad F = \frac{r^2}{1-\lambda r^2}\sum_{i=1}^{n}\frac{r_i^2}{r^2+\xi_i^2}, \qquad U = \sum_{i=1}^{n}\frac{r_i^2}{r^2+\xi_i^2}\prod_{j=1}^{n}(r^2+\xi_j^2)^{m_j+1}. \tag{C.18}$$

The constraint (C.17) can be solved by setting

$$r_i^2 = \prod_{\mu=1}^{n-1}(\xi_i^2 - x_\mu^2)\prod_{\substack{j=1\\(j\neq i)}}^{n}(\xi_i^2-\xi_j^2)^{-1}, \qquad i = 1, 2, \ldots, n. \tag{C.19}$$

Then the above functions turn into the following forms:

$$W = \prod_{\mu=1}^{n-1}(1+\lambda x_\mu^2)\prod_{i=1}^{n}(1+\lambda\xi_i^2)^{-1}, \qquad F = r^2(1-\lambda r^2)^{-1}\prod_{\mu=1}^{n-1}(r^2+x_\mu^2)\prod_{i=1}^{n}(r^2+\xi_i^2)^{-1}, \tag{C.20}$$

$$U = \prod_{\mu=1}^{n-1}(r^2+x_\mu^2)\prod_{i=1}^{n}(r^2+\xi_i^2)^{m_i}. \tag{C.21}$$

One can check that

$$\sum_{i=1}^{n}\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right)^2 dr_i^2+\frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+\xi_i^2)r_i dr_i}{1+\lambda\xi_i^2}\right)^2=\sum_{\mu=1}^{n-1}\frac{dx_\mu^2}{P_\mu}, \tag{C.22}$$

where (for $\mu=1,2,\dots,n-1$)

$$P_\mu:=-\frac{(1+\lambda x_\mu^2)}{x_\mu^2 U_\mu}\prod_{i=1}^{n}(x_\mu^2-\xi_i^2), \qquad U_\mu=(x_\mu^2+r^2)\prod_{\substack{\nu=1\\(\nu\neq\mu)}}^{n-1}(x_\mu^2-x_\nu^2). \tag{C.23}$$

It is convenient to introduce coordinates $(\tilde{t},\tilde{\psi}_i,\tilde{r})$ by

$$\begin{aligned} d\tilde{t} &:= dt-\frac{2Mr^2}{(1-\lambda r^2)V(r)}dr,\\ d\tilde{\psi}_i &= d\psi_i-\frac{2M\xi_i r^2}{(r^2+\xi_i^2)V(r)}dr,\\ d\tilde{r} &:= dr, \end{aligned} \tag{C.24}$$

where

$$V(r)=-2Mr^2+(1-\lambda r^2)\prod_{i=1}^{n}(r^2+\xi_i^2)^{m_i+1}. \tag{C.25}$$

With some work, we can prove that the following relation holds:

$$\begin{aligned} &Fdr^2-W(1-\lambda r^2)dt^2+\sum_{i=1}^{n}\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right)r_i^2(d\psi_i-2A_i)^2\\ &+\frac{2M}{U}\left(Wdt-\sum_{i=1}^{n}\left(\frac{\xi_i r_i^2}{1+\lambda\xi_i^2}\right)(d\psi_i-2A_i)+Fdr\right)^2\\ &=-\frac{d\tilde{r}^2}{P_n}-W(1-\lambda\tilde{r}^2)d\tilde{t}^{\,2}+\sum_{i=1}^{n}\left(\frac{\tilde{r}^2+\xi_i^2}{1+\lambda\xi_i^2}\right)r_i^2(d\tilde{\psi}_i-2A_i)^2\\ &+\frac{2M}{U}\left(Wd\tilde{t}-\sum_{i=1}^{n}\left(\frac{\xi_i r_i^2}{1+\lambda\xi_i^2}\right)(d\tilde{\psi}_i-2A_i)\right)^2, \end{aligned} \tag{C.26}$$

where

$$P_n=\frac{\tilde{X}_n(\tilde{r})}{U_n}, \tag{C.27}$$

$$\tilde{X}_n(\tilde{r})=(-1)^n V(\tilde{r})\tilde{r}^{-2}\prod_{i=1}^{n}(\tilde{r}^2+\xi_i^2)^{-m_i}, \qquad U_n=(-1)^{n-1}\prod_{\mu=1}^{n-1}(\tilde{r}^2+x_\mu^2). \tag{C.28}$$

We rescale the Fubini-Study metric such that

$$\left(\frac{r^2+\xi_i^2}{1+\lambda\xi_i^2}\right) r_i^2 d\Sigma^2_{i,(m_i)} = \prod_{\mu=1}^{n}(x_\mu^2-\xi_i^2)g^{(i)}, \qquad i=1,2,\dots,n, \tag{C.29}$$

with $\tilde{r}=ix_n$, namely

$$g^{(i)} := (-1)^n(1+\lambda\xi_i^2)^{-1}\prod_{\substack{j=1\\(j\neq i)}}^{n}(\xi_i^2-\xi_j^2)^{-1}d\Sigma^2_{i,(m_i)}. \tag{C.30}$$

The Fubini-Study metric $g^{(i)}$ on $\mathbb{CP}^{m_i}$ has the cosmological constant

$$\lambda^{(i)} = (-1)^n 2(m_i+1)(1+\lambda\xi_i^2)\prod_{\substack{j=1\\(j\neq i)}}^{n}(\xi_i^2-\xi_j^2). \tag{C.31}$$

We also rescale the corresponding potential A_i and the angle $\tilde{\psi}_i$ as follows:

$$\tilde{\psi}_i' := (-1)^n(1+\lambda\xi_i^2)^{-1}\prod_{\substack{j=1\\(j\neq i)}}^{n}(\xi_i^2-\xi_j^2)^{-1}\tilde{\psi}_i, \tag{C.32}$$

$$A_i' := (-1)^n(1+\lambda\xi_i^2)^{-1}\prod_{\substack{j=1\\(j\neq i)}}^{n}(\xi_i^2-\xi_j^2)^{-1}A_i. \tag{C.33}$$

For simplicity, let

$$\xi_0^2 := -\frac{1}{\lambda}, \qquad d\tilde{t} =: -\frac{1}{\xi_0}\prod_{i=1}^{n}(\xi_i^2-\xi_0^2)d\tilde{\psi}_0', \tag{C.34}$$

$$\vartheta_0 := d\tilde{\psi}_0', \qquad \vartheta_i := d\tilde{\psi}_i' - 2A_i', \qquad i=1,2,\dots,n. \tag{C.35}$$

We can check that

$$\begin{aligned}
&-W(1-\lambda\tilde{r}^2)d\tilde{t}^{\,2} + \sum_{i=1}^{n}\left(\frac{\tilde{r}^2+\xi_i^2}{1+\lambda\xi_i^2}\right) r_i^2(d\tilde{\psi}_i-2A_i)^2\\
&+\frac{2M}{U}\left(W d\tilde{t} - \sum_{i=1}^{n}\left(\frac{\xi_i r_i^2}{1+\lambda\xi_i^2}\right)(d\tilde{\psi}_i-2A_i)\right)^2\\
&=\sum_{\mu=1}^{n}P_\mu\left(\sum_{\hat{i}=0}^{n}\xi_{\hat{i}}\prod_{\substack{\nu=1\\(\nu\neq\mu)}}^{n}(x_\nu^2-\xi_{\hat{i}}^2)\vartheta_{\hat{i}}\right)^2 + \frac{c}{\sigma_n}\left(\sum_{\hat{i}=0}^{n}\frac{1}{\xi_{\hat{i}}}\prod_{\nu=1}^{n}(x_\nu^2-\xi_{\hat{i}})\vartheta_{\hat{i}}\right)^2,
\end{aligned} \tag{C.36}$$

where

$$c = -\prod_{i=1}^{n} \xi_i^2. \tag{C.37}$$

Let us define 1-forms θ_k by

$$\theta_k := \sum_{\hat{i}=0}^{n} (-1)^{n-k} \xi_{\hat{i}}^{2(n-k)-1} \vartheta_{\hat{i}}, \qquad k = 0, 1, \ldots, n. \tag{C.38}$$

They satisfy the equations (2.6) for $N = n$ and $\varepsilon = 1$:

$$d\theta_k + 2\sum_{i=1}^{n} (-1)^{n-k} \xi_i^{2(n-k)-1} \omega^{(i)} = 0, \qquad \omega^{(i)} := dA_i', \qquad (k = 0, 1, \ldots, n). \tag{C.39}$$

Note that

$$\begin{aligned} \theta_k &= \sum_{\hat{i}=0}^{n} (-1)^{n-k} \xi_{\hat{i}}^{2(n-k)-1} \vartheta_{\hat{i}} \\ &= -\frac{\lambda^k}{\prod_{i=1}^{n}(1+\lambda\xi_i^2)} d\tilde{t} + (-1)^{n-k} \sum_{i=1}^{n} \xi_i^{2(n-k)-1} (d\tilde{\psi}_i' - 2A_i') \\ &= -\frac{\lambda^k}{\prod_{i=1}^{n}(1+\lambda\xi_i^2)} d\tilde{t} + (-1)^k \sum_{i=1}^{n} \frac{\xi_i^{2(n-k)-1}}{(1+\lambda\xi_i^2) \prod_{\substack{j=1 \\ (j\neq i)}}^{n} (\xi_i^2 - \xi_j^2)} (d\tilde{\psi}_i - 2A_i). \end{aligned} \tag{C.40}$$

Using these 1-forms θ_k, summations within brackets in the last line of (C.36) can be rewritten as follows:

$$\sum_{\hat{i}=0}^{n} \xi_{\hat{i}} \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} (x_\nu^2 - \xi_{\hat{i}}^2) \vartheta_{\hat{i}} = -\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k, \tag{C.41}$$

$$\sum_{\hat{i}=0}^{n} \frac{1}{\xi_{\hat{i}}} \prod_{\nu=1}^{n} (x_\nu^2 - \xi_{\hat{i}}) \vartheta_{\hat{i}} = \sum_{k=0}^{n} \sigma_k \theta_k. \tag{C.42}$$

Combining these relations, the metric (C.16) finally becomes the metric (3.9):

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k \right]^2 + \frac{c}{\sigma_n} \left[\sum_{k=0}^{n} \sigma_k \theta_k \right]^2 + \sum_{i=1}^{n} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_i^2) g^{(i)}, \tag{C.43}$$

where

$$P_\mu = \frac{X_\mu(x_\mu)}{x_\mu \prod_{i=1}^{n}(x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} (x_\mu^2 - x_\nu^2), \qquad c = -\prod_{i=1}^{n} \xi_i^2, \tag{C.44}$$

$$X_\mu(x_\mu) = x_\mu \left((-1)^{n+|m|-1} 2M\delta_{\mu,n} - \left(1 + \lambda x_\mu^2\right) x_\mu^{-2} \prod_{i=1}^{n} (x_\mu^2 - \xi_i^2)^{m_i+1} \right). \tag{C.45}$$

Hence, we have seen that by the coordinate transformations (C.7), (C.9), (C.12), (C.19) and (C.24), the odd-dimensional general Kerr-de Sitter metric (C.1) with equal angular momenta turns into the subfamily of the special case of the generalized Kerr-NUT-de Sitter metric (3.9). Here all angular momenta take non-zero values.

Let us summarize the special type metric (3.9) with $N = n$, $m^{(0)} = 0$ and $\varepsilon = 1$: the base space is the direct product of complex projective spaces

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(n)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_n} \tag{C.46}$$

and the fiber over B is a $(2n+1)$-dimensional Kerr-de Sitter space.

C.1 From special type to odd dimensional general type

By sending one of constant eigenvalues, say ξ_n, to zero, the metric of special type (3.9) goes to a metric of general type with $N = n-1$, $m^{(0)} = 2m_n + 1$ and $\varepsilon = 0$. The base space is the direct product of complex projective spaces and a $(2m_n+1)$-dimensional sphere S^{2m_n+1}

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(n-1)} \times M^{(0)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_{n-1}} \times S^{2m_n+1}, \tag{C.47}$$

and the fiber over B is a $2n$-dimensional Kerr-de Sitter space.

For $k = 0, 1, \ldots, n-1$, the 1-form θ_k (C.40) has a smooth $\xi_n \to 0$ limit:

$$\begin{aligned}\tilde{\theta}_k &:= \lim_{\xi_n \to 0} \theta_k \\ &= -\lambda^k \prod_{i=1}^{n-1} (1+\lambda\xi_j^2)^{-1} d\tilde{t} \\ &+ (-1)^k \sum_{i=1}^{n-1} \xi_i^{2(n-1-k)-1} (1+\lambda\xi_i^2)^{-1} \prod_{\substack{j=1 \\ (j\neq i)}}^{n-1} (\xi_i^2 - \xi_j^2)^{-1} (d\tilde{\psi}_i - 2A_i),\end{aligned} \tag{C.48}$$

while θ_n has a singular limit

$$\theta_n = -\frac{1}{\xi_n} \prod_{j=1}^{n-1} \xi_j^{-2} (d\psi_n - 2A_n) + O(1). \tag{C.49}$$

Here we have used

$$d\tilde{\psi}_n = d\psi_n + O(\xi_n). \tag{C.50}$$

The leading term of $\sum_{k=0}^n \sigma_k \theta_k$ is $\sigma_n \theta_n$. Hence it follows that

$$\lim_{\xi_n \to 0} \frac{c}{\sigma_n} \left[\sum_{k=0}^n \sigma_k \theta_k \right]^2 = -\sigma_n \left(\prod_{j=1}^{n-1} \xi_j^{-2} \right) (d\psi_n - 2A_n)^2. \tag{C.51}$$

Also

$$\lim_{\xi_n \to 0} g^{(n)} = -\left(\prod_{j=1}^{n-1} \xi_j^{-2} \right) d\Sigma^2_{n,(m_n)}. \tag{C.52}$$

In the $\xi_n \to 0$ limit, the metric (3.9) becomes

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^n P_\mu \left[\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \tilde{\theta}_k \right]^2 + \sum_{i=1}^{n-1} \prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) g^{(i)} - \sigma_n \left(\prod_{j=1}^{n-1} \xi_j^{-2} \right) d\Omega^2_{n,(2m_n+1)}, \tag{C.53}$$

where

$$d\Omega^2_{n,(2m_n+1)} = (d\psi_n - 2A_n)^2 + d\Sigma^2_{n,(m_n)}, \tag{C.54}$$

$$P_\mu = \frac{X_\mu(x_\mu)}{(x_\mu)^{2m_n+1} \prod_{i=1}^{n-1} (x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \tag{C.55}$$

$$X_\mu(x_\mu) = x_\mu \left((-1)^{(1/2)(D-1)-1} 2M\delta_{\mu,n} - \left(1+\lambda x_\mu^2\right) {x_\mu}^{2m_n} \prod_{i=1}^{n-1} (x_\mu^2 - \xi_i^2)^{m_i+1} \right). \tag{C.56}$$

Therefore, we get the metric (3.18) for odd $m^{(0)}$. Hence we have shown that the odd dimensional case of (3.18) represents the odd dimensional Kerr-de Sitter black hole with partially equal angular momenta and with some zero angular momenta.

Remark. The function $\chi(x) = \sum_{i=-1}^{n} \alpha_i x^{2i}$ (3.13) has a smooth limit into $\chi(x) = \sum_{i=0}^{n} \alpha_i x^{2i}$: $\lim_{\xi_n \to 0} \alpha_{-1} = 0$, and

$$
\begin{aligned}
\chi(x) = \sum_{i=0}^{n} \alpha_i x^{2i} &= -2 \sum_{i=1}^{n-1} (m_i + 1)(1 + \lambda \xi_i^2) x^2 \prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} (x^2 - \xi_j^2) \\
&\quad - 2\Big(m_n (1 + \lambda x^2) + (n + |m|) \lambda x^2 \Big) \prod_{i=1}^{n-1} (x^2 - \xi_i^2),
\end{aligned} \tag{C.57}
$$

where

$$
\alpha_0 = (-1)^n 2 m_n \prod_{j=1}^{n-1} \xi_j^2, \qquad |m| = \sum_{i=1}^{n-1} m_i. \tag{C.58}
$$

D $D = 2n'$ Kerr-de Sitter black hole

The general Kerr-de Sitter metric in $D = 2n'$ can be obtained from that of $D = 2n' + 1$ by setting [16, 17]

$$
\phi_{n'} = 0, \qquad a_{n'} = 0, \qquad M \to Mr. \tag{D.1}
$$

The metric is given by

$$
g = d\bar{s}^2 + \frac{2Mr}{U} \left(W dt + F dr - \sum_{I=1}^{n'-1} \frac{a_I \mu_I^2}{1 + \lambda a_I^2} d\phi_I \right)^2, \tag{D.2}
$$

$$
\begin{aligned}
d\bar{s}^2 = &-W(1 - \lambda r^2) dt^2 + F dr^2 + \sum_{I=1}^{n'} \frac{r^2 + a_I^2}{1 + \lambda a_I^2} d\mu_I^2 + \sum_{I=1}^{n'-1} \frac{r^2 + a_I^2}{1 + \lambda a_I^2} \mu_I^2 d\phi_I^2 \\
&+ \frac{\lambda}{W(1 - \lambda r^2)} \left(\sum_{I=1}^{n'} \frac{(r^2 + a_I^2) \mu_I d\mu_I}{1 + \lambda a_I^2} \right)^2,
\end{aligned} \tag{D.3}
$$

$$
W = \sum_{I=1}^{n'} \frac{\mu_I^2}{1 + \lambda a_I^2}, \qquad F = \frac{r^2}{1 - \lambda r^2} \sum_{I=1}^{n'} \frac{\mu_I^2}{r^2 + a_I^2}, \tag{D.4}
$$

$$U = r^2 \left(\sum_{I=1}^{n'} \frac{\mu_I^2}{r^2 + a_I^2} \right) \prod_{J=1}^{n'-1} (r^2 + a_J^2), \qquad \sum_{I=1}^{n'} \mu_I^2 = 1. \tag{D.5}$$

Remark. For $D = 2n'$, U/r here is written as U in [16, 17].

Then taking

$$n' = n + |m| = n + \sum_{i=1}^{n} m_i, \qquad m_n := 0, \qquad \xi_n = 0, \tag{D.6}$$

with (D.1), the metric (D.2) with partially equal angular momenta can be written as

$$\begin{aligned} g = &\sum_{i=1}^{n} \left(\frac{r^2 + \xi_i^2}{1 + \lambda \xi_i^2} \right) dr_i^2 + \frac{\lambda}{W(1 - \lambda r^2)} \left(\sum_{i=1}^{n} \frac{(r^2 + \xi_i^2) r_i dr_i}{1 + \lambda \xi_i^2} \right)^2 \\ &+ F dr^2 - W(1 - \lambda r^2) dt^2 + \sum_{i=1}^{n-1} \left(\frac{r^2 + \xi_i^2}{1 + \lambda \xi_i^2} \right) r_i^2 (d\psi_i - 2A_i)^2 \\ &+ \frac{2Mr}{U} \left(W dt - \sum_{i=1}^{n-1} \left(\frac{\xi_i r_i^2}{1 + \lambda \xi_i^2} \right) (d\psi_i - 2A_i) + F dr \right)^2 \\ &+ \sum_{i=1}^{n-1} \left(\frac{r^2 + \xi_i^2}{1 + \lambda \xi_i^2} \right) r_i^2 d\Sigma_{i,(m_i)}^2, \end{aligned} \tag{D.7}$$

with a constraint

$$\sum_{i=1}^{n} r_i^2 = 1. \tag{D.8}$$

This can be rewritten as the form

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k \right)^2 + \sum_{i=1}^{n-1} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_i^2) g^{(i)}, \tag{D.9}$$

by the coordinate transformations:

$$\begin{aligned} d\tilde{t} &= dt - \frac{2Mr^3}{(1 - \lambda r^2) V(r)} dr, \\ d\tilde{\psi}_i &= d\psi_i - \frac{2M \xi_i r^3}{(r^2 + \xi_i^2) V(r)} dr, \\ d\tilde{r} &= dr, \end{aligned} \tag{D.10}$$

$$V(r) = -2Mr^3 + (1 - \lambda r^2) r^2 \prod_{i=1}^{n-1} (r^2 + \xi_i^2)^{m_i + 1}, \tag{D.11}$$

$$r_i^2 = \prod_{\mu=1}^{n-1}(\xi_i^2 - x_\mu^2) \prod_{\substack{j=1 \\ (j\neq i)}}^{n} (\xi_i^2 - \xi_j^2)^{-1}, \qquad i = 1, 2, \ldots, n, \qquad \tilde{r} = i x_n, \tag{D.12}$$

$$g^{(i)} = (-1)^n (1 + \lambda\xi_i^2)^{-1} \xi_i^{-2} \prod_{\substack{j=1 \\ (j\neq i)}}^{n-1} (\xi_i^2 - \xi_j^2)^{-1} d\Sigma^2_{i,(m_i)}, \qquad i = 1, 2, \ldots, n-1, \tag{D.13}$$

$$\xi_0^2 := -\frac{1}{\lambda}, \tag{D.14}$$

$$\begin{aligned}
\theta_k &= \sum_{\hat{i}=0}^{n-1}(-1)^{n-k} \xi_{\hat{i}}^{2(n-k)-1} \vartheta_{\hat{i}} \\
&= -\lambda^k \prod_{j=1}^{n-1}(1+\lambda\xi_j^2)^{-1} d\tilde{t} + (-1)^{n-k} \sum_{i=1}^{n-1} \xi_i^{2(n-k)-1}(d\tilde{\psi}_i' - 2A_i') \\
&= -\lambda^k \prod_{j=1}^{n-1}(1+\lambda\xi_j^2)^{-1} d\tilde{t} + (-1)^{k} \sum_{i=1}^{n-1} \xi_i^{2(n-1-k)-1}(1+\lambda\xi_i^2)^{-1} \prod_{\substack{j=1 \\ (j\neq i)}}^{n-1} (\xi_i^2 - \xi_j^2)^{-1}(d\tilde{\psi}_i - 2A_i).
\end{aligned} \tag{D.15}$$

Here (for $\mu = 1, 2, \ldots, n$)

$$P_\mu = \frac{X_\mu(x_\mu)}{\prod_{i=1}^{n-1}(x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} (x_\mu^2 - x_\nu^2), \tag{D.16}$$

$$X_\mu(x_\mu) = (-1)^{n+|m|-1} 2Mi x_\mu \delta_{\mu,n} - (1+\lambda x_\mu^2) \prod_{j=1}^{n-1}(x_\mu^2 - \xi_j^2)^{m_j+1}. \tag{D.17}$$

This metric (D.9) represents the general type with $m^{(0)} = 0$, $\varepsilon = 0$,

$$D = 2n + 2|m| = 2n + 2\sum_{i=1}^{n-1} m_i, \tag{D.18}$$

the base space is ($N = n - 1$)

$$B = M^{(1)} \times M^{(2)} \times \cdots \times M^{(n-1)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_{n-1}}, \tag{D.19}$$

and the fiber is a $2n$-dimensional Kerr-de Sitter spacetime.

The Fubini-Study metric $g^{(i)}$ $(i = 1, 2, \dots, n-1)$ has the cosmological constant

$$\lambda^{(i)} = (-1)^n 2(m_i + 1)(1 + \lambda\xi_i^2)\xi_i^2 \prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} (\xi_i^2 - \xi_j^2). \tag{D.20}$$

D.1 Introducing $M^{(0)}$ by taking zero eigenvalue limit

The base space (D.19) has no Einstein subspace $M^{(0)}$ which corresponds to the zero eigenvalues of the CKY tensor. In this subsection, we introduce $M^{(0)}$ by sending one of non-zero constant eigenvalues ξ_i to zero.

Let us consider a limit such that one of the constant eigenvalues, say ξ_{n-1}, goes to zero. Since $\xi_n = 0$, naive $\xi_{n-1} \to 0$ limit make the coordinate transformation (D.12) singular. It is convenient to rescale one of x_μ. For definiteness, we set $x_{n-1} = \xi_{n-1}\rho$ and take $\xi_{n-1} \to 0$ limit keeping ρ finite. The coordinate transformation (D.12) becomes

$$\begin{aligned} r_i^2 &= \xi_i^{-2} \prod_{\mu=1}^{n-1} (\xi^2 - x_\mu^2) \left(\prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} (\xi_i^2 - \xi_j^2)^{-1} \right), \qquad i = 1, 2, \dots, n-2, \\ r_{n-1}^2 &= \prod_{\mu=1}^{n-2} x_\mu^2 \left(\prod_{j=1}^{n-2} \xi_j^{-2} \right) (1 - \rho^2), \\ r_n^2 &= \prod_{\mu=1}^{n-2} x_\mu^2 \left(\prod_{j=1}^{n-2} \xi_j^{-2} \right) \rho^2. \end{aligned} \tag{D.21}$$

The range of ρ is restricted to the interval $-1 \leq \rho \leq 1$. We can see that

$$\lim_{\xi_{n-1} \to 0} \frac{P_{n-1}}{\xi_{n-1}^2} = -\frac{1}{\tilde{\sigma}_{n-1}} \prod_{i=1}^{n-2} \xi_i^2 (1 - \rho^2), \qquad \tilde{\sigma}_{n-1} := \left(\prod_{\mu=1}^{n-2} x_\mu^2 \right) x_n^2. \tag{D.22}$$

$$\lim_{\xi_{n-1} \to 0} P_{n-1} \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_{n-1}) \theta_k \right)^2 = -\tilde{\sigma}_{n-1} \left(\prod_{j=1}^{n-2} \xi_j^{-2} \right) (1 - \rho^2)(d\psi_{n-1} - 2A_{n-1})^2, \tag{D.23}$$

$$\lim_{\xi_{n-1} \to 0} \left(\frac{r^2 + \xi_{n-1}^2}{1 + \lambda\xi_{n-1}^2} \right) r_{n-1}^2 d\Sigma_{n-1,(m_{n-1})}^2 = -\tilde{\sigma}_{n-1} \left(\prod_{j=1}^{n-2} \xi_j^{-2} \right) (1 - \rho^2) d\Sigma_{n-1,(m_{n-1})}^2. \tag{D.24}$$

Combining these relations, we have

$$\begin{aligned}&\frac{dx_{n-1}^2}{P_{n-1}}+P_{n-1}\left(\sum_{k=0}^{n-1}\sigma_k(\hat{x}_{n-1})\theta_k\right)^2+\left(\frac{r^2+\xi_{n-1}^2}{1+\lambda\xi_{n-1}^2}\right)r_{n-1}^2d\Sigma_{n-1,(m_{n-1})}^2\\&\to-\tilde{\sigma}_{n-1}\left(\prod_{j=1}^{n-2}\xi_j^{-2}\right)d\tilde{\Omega}_{n-1,(2m_{n-1}+2)}^2,\end{aligned}\tag{D.25}$$

where

$$d\tilde{\Omega}_{n-1,(2m_{n-1}+2)}^2:=\frac{d\rho^2}{1-\rho^2}+(1-\rho^2)d\Omega_{n-1,(2m_{n-1}+1)}^2\tag{D.26}$$

is the metric on the sphere $S^{2m_{n-1}+2}$ with unit radius.

For $\mu=1,2,\dots,n-2,n$, let

$$\prod_{\substack{\nu=1\\(\nu\neq\mu,n-1)}}^{n}(t-x_\nu^2)=\sum_{k=0}^{n-2}(-1)^kt^{n-2-k}\sigma_k(\hat{x}_\mu,\hat{x}_{n-1})=:\sum_{k=0}^{n-2}(-1)^kt^{n-2-k}\tilde{\sigma}_k(\hat{x}_\mu).\tag{D.27}$$

It holds that for $\mu\neq n-1$,

$$\sigma_k(\hat{x}_\mu)=\tilde{\sigma}_k(\hat{x}_\mu)+x_{n-1}^2\tilde{\sigma}_{k-1}(\hat{x}_\mu).\tag{D.28}$$

Especially, for $k=n-1$, we have

$$\sigma_{n-1}(\hat{x}_\mu)=x_{n-1}^2\tilde{\sigma}_{n-2}(\hat{x}_\mu)=\xi_{n-1}^2\rho^2\tilde{\sigma}_{n-2}(\hat{x}_\mu)=O(\xi_{n-1}^2).\tag{D.29}$$

From the last expression of θ_k in (D.15), we can see that $\theta_k=O(1)$ for $k=0,1,\dots,n-2$ and $\theta_{n-1}=O(\xi_{n-1}^{-1})$. We have

$$\lim_{\xi_{n-1}\to 0}\sigma_{n-1}(\hat{x}_\mu)\theta_{n-1}=0,\qquad \mu\neq n-1,\tag{D.30}$$

and

$$\lim_{\xi_{n-1}\to 0}\sum_{k=0}^{n-1}\sigma_k(\hat{x}_\mu)\theta_k=\sum_{k=0}^{n-2}\tilde{\sigma}_k(\hat{x}_\mu)\tilde{\theta}_k,\qquad \mu\neq n-1,\tag{D.31}$$

where

$$\tilde{\theta}_k=-\lambda^k\prod_{j=1}^{n-2}(1+\lambda\xi_j^2)^{-1}d\tilde{t}+(-1)^k\sum_{i=1}^{n-2}\xi_i^{2(n-2-k)-1}(1+\lambda\xi_i^2)^{-1}\prod_{\substack{j=1\\(j\neq i)}}^{n-2}(\xi_i^2-\xi_j^2)^{-1}(d\tilde{\psi}_i-2A_i).\tag{D.32}$$

For $\mu \neq n-1$,

$$P_\mu = \frac{X_\mu(x_\mu)}{(x_\mu)^{2m_{n-1}+2} \prod_{i=1}^{n-2} (x_\mu^2 - \xi_i^2) \tilde{U}_\mu}, \qquad \tilde{U}_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu, n-1)}}^{n} (x_\mu^2 - x_\nu^2), \tag{D.33}$$

$$X_\mu(x_\mu) = (-1)^{(1/2)D} 2Mi x_\mu \delta_{\mu,n} - (1+\lambda x_\mu^2)(x_\mu)^{2m_{n-1}+2} \prod_{i=1}^{n-2} (x_\mu^2 - \xi_i^2)^{m_i+1}. \tag{D.34}$$

Note that

$$g^{(i)} = (-1)^n (1+\lambda \xi_i^2)^{-1} \xi_i^{-2} \prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} (\xi_i^2 - \xi_j^2)^{-1} d\Sigma_{i,(m_i)}^2. \tag{D.35}$$

For $i \neq n-1$,

$$\tilde{g}^{(i)} := -\xi_i^2 \lim_{\xi_{n-1} \to 0} g^{(i)} = (-1)^{n-1} (1+\lambda \xi_i^2)^{-1} \xi_i^{-2} \prod_{\substack{j=1 \\ (j \neq i)}}^{n-2} (\xi_i^2 - \xi_j^2)^{-1} d\Sigma_{i,(m_i)}^2. \tag{D.36}$$

In the $\xi_{n-1} \to 0$ limit, the metric (3.18) becomes

$$\begin{aligned} g = & \sum_{\substack{\mu=1 \\ (\mu \neq n-1)}}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\substack{\mu=1 \\ (\mu \neq n-1)}}^{n} P_\mu \left[\sum_{k=0}^{n-2} \tilde{\sigma}_k(\hat{x}_\mu) \tilde{\theta}_k \right]^2 \\ & + \sum_{i=1}^{n-2} \prod_{\substack{\mu=1 \\ (\mu \neq n-1)}}^{n} (x_\mu^2 - \xi_i^2) \tilde{g}^{(i)} - \hat{\sigma}_{n-1} \left(\prod_{j=1}^{n-2} \xi_j^{-2} \right) d\tilde{\Omega}_{n-1,(2m_{n-1}+2)}^2, \end{aligned} \tag{D.37}$$

with

$$N = n-2, \qquad m^{(0)} = 2m_{n-1}+2, \qquad \varepsilon = 0. \tag{D.38}$$

The base space is given by

$$B = M^{(1)} \times M^{(2)} \times \cdots M^{(n-2)} \times M^{(0)} = \mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2} \times \cdots \times \mathbb{CP}^{m_{n-2}} \times S^{2m_{n-1}+2}, \tag{D.39}$$

and the fiber over B is a $2(n-1)$-dimensional Kerr-de Sitter space.

By replacing $n-1$ with n, we obtain (3.18) for even $m^{(0)}$. Hence we have shown that the even dimensional case of (3.18) represents the even dimensional Kerr-de Sitter black hole with partially equal angular momenta and with some zero angular momenta.

E Lichnerowicz operator

E.1 General type

We evaluate the Lichnerowicz operator $\Delta_L h_{AB}$. The non-zero components are calculated as follows:

- **vector**

$$\Delta_L h_{n+\mu,(\hat{\alpha},i)} = \frac{2\xi_i\sqrt{P_\mu}}{x_\mu^2-\xi_i^2}\left(\prod_{\nu=1}^{n}(x_\nu^2-\xi_i^2)\right)^{-1/2}(\mathcal{D}^{(i)}_\beta h_{(m_i+\beta,i),(\hat{\alpha},i)} - \mathcal{D}^{(i)}_{m_i+\beta} h_{(\beta,i),(\hat{\alpha},i)}), \tag{E.1}$$

- **tensor**

$$\begin{aligned}
\Delta_L\, h_{(\alpha,i),(\beta,i)} &= \Box^{(F)} h_{(\alpha,i),(\beta,i)} + 2\sum_\mu \frac{\xi_i\sqrt{P_\mu}}{x_\mu^2-\xi_i^2} e_{n+\mu}(h_{(m_i+\alpha,i),(\beta,i)} + h_{(\alpha,i),(m_i+\beta,i)}) \\
&- \sum_{(\hat{\gamma},j)(j\neq i)} \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_j^2)} (\mathcal{D}^{(j)}_{\hat{\gamma}} \bar{e}^{(j)}_{\hat{\gamma}} h_{(\alpha,i),(\beta,i)}) - 2\sum_{j,\mu} \frac{m_j x_\mu\sqrt{P_\mu}}{x_\mu^2-\xi_j^2}(e_\mu h_{(\alpha,i),(\beta,i)}) \\
&- \sum_a \frac{1}{\sigma_n}(D^{(0)}_a \tilde{e}_a h_{(\alpha,i),(\beta,i)}) - \sum_\mu \frac{m^{(0)}\sqrt{P_\mu}}{x_\mu}(e_\mu h_{(\alpha,i),(\beta,i)}) \\
&- \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_i^2)}\left(\sum_{\hat{\gamma}} \mathcal{D}^{(i)}_{\hat{\gamma}} \mathcal{D}^{(i)}_{\hat{\gamma}} h_{(\alpha,i),(\beta,i)} + 2\sum_{\hat{\gamma},\hat{\delta}} \tilde{R}^{(i)}_{\alpha\hat{\gamma}\beta\hat{\delta}} h_{(\hat{\gamma},i),(\hat{\delta},i)}\right) \\
&+ 4\sum_\mu \frac{\xi_i^2 P_\mu}{(x_\mu^2-\xi_i^2)^2} h_{(\alpha,i),(\beta,i)} + 4\sum_{\mu,j} \frac{\xi_i\xi_j P_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)} h_{(m_i+\alpha,i),(m_i+\beta,i)} \\
&+ 2\Lambda h_{(\alpha,i),(\beta,i)},
\end{aligned} \tag{E.2}$$

$$
\begin{aligned}
\Delta_L\, h_{(\alpha,i),(m_i+\beta,i)} &= \Box^{(F)} h_{(\alpha,i),(m_i+\beta,i)} + 2\sum_{\mu} \frac{\xi_i\sqrt{P_\mu}}{x_\mu^2-\xi_i^2} e_{n+\mu}\big(h_{(m_i+\alpha,i),(m_i+\beta,i)} - h_{(\alpha,i),(\beta,i)}\big) \\
&- \sum_{(\hat\gamma,j)(j\neq i)} \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_j^2)} \big(\mathcal{D}^{(j)}_{\hat\gamma}\bar e^{(j)}_{\hat\gamma} h_{(\alpha,i),(m_i+\beta,i)}\big) - 2\sum_{j,\mu} \frac{m_j x_\mu \sqrt{P_\mu}}{x_\mu^2-\xi_j^2}\big(e_\mu h_{(\alpha,i),(m_i+\beta,i)}\big) \\
&- \sum_a \frac{1}{\sigma_n}\big(D^{(0)}_a \tilde e_a h_{(\alpha,i),(m_i+\beta,i)}\big) - \sum_\mu \frac{m^{(0)}\sqrt{P_\mu}}{x_\mu}\big(e_\mu h_{(\alpha,i),(m_i+\beta,i)}\big) \\
&- \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_i^2)} \left(\sum_{\hat\gamma} \mathcal{D}^{(i)}_{\hat\gamma}\mathcal{D}^{(i)}_{\hat\gamma} h_{(\alpha,i),(m_i+\beta,i)} + 2\sum_{\hat\gamma,\hat\delta} \tilde R^{(i)}_{\alpha,\hat\gamma,m_i+\beta,\hat\delta} h_{(\hat\gamma,i),(\hat\delta,i)} \right) \\
&+ 4\sum_\mu \frac{\xi_i^2 P_\mu}{(x_\mu^2-\xi_i^2)^2} h_{(\alpha,i),(m_i+\beta,i)} - 4\sum_{\mu,j} \frac{\xi_i\xi_j P_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)} h_{(m_i+\alpha,i),(\beta,i)} \\
&+ 2\Lambda h_{(\alpha,i),(m_i+\beta,i)},
\end{aligned} \tag{E.3}
$$

$$
\begin{aligned}
\Delta_L\, h_{(m_i+\alpha,i),(m_i+\beta,i)} &= \Box^{(F)} h_{(m_i+\alpha,i),(m_i+\beta,i)} - 2\sum_{\mu} \frac{\xi_i\sqrt{P_\mu}}{x_\mu^2-\xi_i^2} e_{n+\mu}\big(h_{(\alpha,i),(m_i+\beta,i)} + h_{(m_i+\alpha,i),(\beta,i)}\big) \\
&- \sum_{(\hat\gamma,j)(j\neq i)} \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_j^2)} \big(\mathcal{D}^{(j)}_{\hat\gamma}\bar e^{(j)}_{\hat\gamma} h_{(m_i+\alpha,i),(m_i+\beta,i)}\big) - 2\sum_{j,\mu} \frac{m_j x_\mu \sqrt{P_\mu}}{x_\mu^2-\xi_j^2}\big(e_\mu h_{(m_i+\alpha,i),(m_i+\beta,i)}\big) \\
&- \sum_a \frac{1}{\sigma_n}\big(D^{(0)}_a \tilde e_a h_{(m_i+\alpha,i),(m_i+\beta,i)}\big) - \sum_\mu \frac{m^{(0)}\sqrt{P_\mu}}{x_\mu}\big(e_\mu h_{(m_i+\alpha,i),(m_i+\beta,i)}\big) \\
&- \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_i^2)} \left(\sum_{\hat\gamma} \mathcal{D}^{(i)}_{\hat\gamma}\mathcal{D}^{(i)}_{\hat\gamma} h_{(m_i+\alpha,i),(m_i+\beta,i)} + 2\sum_{\hat\gamma,\hat\delta} \tilde R^{(i)}_{m_i+\alpha,\hat\gamma,m_i+\beta,\hat\delta} h_{(\hat\gamma,i),(\hat\delta,i)} \right) \\
&+ 4\sum_\mu \frac{\xi_i^2 P_\mu}{(x_\mu^2-\xi_i^2)^2} h_{(m_i+\alpha,i),(m_i+\beta,i)} + 4\sum_{\mu,j} \frac{\xi_i\xi_j P_\mu}{(x_\mu^2-\xi_i^2)(x_\mu^2-\xi_j^2)} h_{(\alpha,i),(\beta,i)} \\
&+ 2\Lambda h_{(m_i+\alpha,i),(m_i+\beta,i)},
\end{aligned} \tag{E.4}
$$

$$
\begin{aligned}
\Delta_L\, h_{ab} &= \Box^{(F)} h_{ab} - \sum_{(\hat\alpha,i)} \frac{1}{\prod_{\mu=1}^n (x_\mu^2-\xi_i^2)} \big(\mathcal{D}^{(i)}_{\hat\alpha}\bar e^{(i)}_{\hat\alpha} h_{ab}\big) \\
&- 2\sum_{i,\mu} \frac{m_i x_\mu\sqrt{P_\mu}}{x_\mu^2-\xi_i^2}(e_\mu h_{ab}) - \frac{1}{\sigma_n}\left(\sum_d D^{(0)}_d D^{(0)}_d h_{ab} + 2\sum_{d,e} \tilde R_{adbe} h_{de} \right) \\
&- m^{(0)} \sum_\mu \frac{\sqrt{P_\mu}}{x_\mu}(e_\mu h_{ab}) + 2\Lambda h_{ab},
\end{aligned} \tag{E.5}
$$

where $\square^{(F)}$ represents the scalar Laplacian on the fiber space. Explicitly, putting $(e_{\hat{\mu}}) = (e_\mu, e_{n+\mu})$ we have[2]

$$\square^{(F)} f = -\sum_{\hat{\mu}} e_{\hat{\mu}}(e_{\hat{\mu}} f) + \sum_{\hat{\mu},\hat{\nu}} (e_{\hat{\nu}} f)\omega_{\hat{\nu}\hat{\mu}}(e_{\hat{\mu}}). \tag{E.6}$$

E.2 Special type

The non-zero components of $\Delta_L h_{AB}$ are calculated as follows:

• vector

$$\Delta_L h_{2n+1,(\hat{\alpha},i)} = -\frac{2\sqrt{S}}{\xi_i}\left(\prod_{\nu=1}^{n}(x_\nu^2 - \xi_i^2)\right)^{-1/2} (\mathcal{D}_\beta^{(i)} h_{(m_i+\beta,i),(\hat{\alpha},i)} - \mathcal{D}_{m_i+\beta}^{(i)} h_{(\beta,i),(\hat{\alpha},i)}) \tag{E.7}$$

together with (E.1).

• tensor

In the formulas (E.2), (E.3) and (E.4), the terms in the third line

$$-\sum_a \frac{1}{\sigma_n}(D_a^{(0)}\tilde{e}_a h_{(\cdot,i),(\cdot,i)}) - \sum_\mu \frac{m^{(0)}\sqrt{P_\mu}}{x_\mu}(e_\mu h_{(\cdot,i),(\cdot,i)}) \tag{E.8}$$

are dropped, and the vector fields $\{e_\mu, e_{n+\mu}\}$ are replaced by $\{\hat{e}_\mu, \hat{e}_{n+\mu}\}$. The following new terms are added to the equations:

$$\begin{aligned} &-\frac{2\sqrt{S}}{\xi_i}\hat{e}_{2n+1}(h_{(m_i+\alpha,i),(\beta,i)} + h_{(\alpha,i),(m_i+\beta,i)}) + \frac{4S}{\xi_i^2}h_{(\alpha,i),(\beta,i)} \\ &+4\sum_j \frac{S}{\xi_i\xi_j}h_{(m_i+\alpha,i),(m_i+\beta,i)} \qquad \text{to (E.2)}\ , \end{aligned} \tag{E.9}$$

$$\begin{aligned} &-\frac{2\sqrt{S}}{\xi_i}\hat{e}_{2n+1}(h_{(m_i+\alpha,i),(m_i+\beta,i)} - h_{(\alpha,i),(\beta,i)}) + \frac{4S}{\xi_i^2}h_{(\alpha,i),(m_i+\beta,i)} \\ &-4\sum_j \frac{S}{\xi_i\xi_j}h_{(m_i+\alpha,i),(\beta,i)} \qquad \text{to (E.3)}\ , \end{aligned} \tag{E.10}$$

$$\begin{aligned} &\frac{2\sqrt{S}}{\xi_i}\hat{e}_{2n+1}(h_{(\alpha,i),(m_i+\beta,i)} + h_{(m_i+\alpha,i),(\beta,i)}) + \frac{4S}{\xi_i^2}h_{(m_i+\alpha,i),(m_i+\beta,i)} \\ &+4\sum_j \frac{S}{\xi_i\xi_j}h_{(\alpha,i),(\beta,i)} \qquad \text{to (E.4)}\ . \end{aligned} \tag{E.11}$$

[2] In the special case, we put $(e_{\hat{\mu}}) = (\hat{e}_\mu, \hat{e}_{n+\mu}, \hat{e}_{2n+1})$.

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